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THE UNIVERSITY OF ALBERTA

THE ELASTODYNAMICS OF A QUARTERSPACE

by

© PRATAP NARAYAN SAHAY

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

IN

GEOPHYSICS

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FALL 1986

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Date...Oct. 8, 1986.....

To my mother and father.

Abstract

Elastodynamic problems in media, bounded by nonparallel surfaces (geometries such as a wedge or a cylindrical hole in a halfspace, etc...), are usually nonseparable boundary-value problems. Due to the presence of different level surfaces, the problem of mode conversion (longitudinal to transverse and vice-versa) and excitation of interface waves at the boundaries is of great complexity and leads to nonseparability. Analytical solutions for nonparallel boundary-value problems do not usually exist in closed form and most of the work done in this area has been limited to very specific aspects of wave propagation rather than being directed towards developing solutions for the full elastic field. This is primarily because of the failure of the existing mathematical techniques for the elastic case which, however, work successfully for the acoustic analogue.

If, for a given geometry and a set of boundary conditions, the wave operator is self-adjoint, then a complete set of eigenfunctions should exist. Using this set as basis, the full elastic field for any source function can be expressed following standard mathematical procedures. Thus, one obtains an exact theory for a nonseparable boundary-value problem.

In this thesis a complete set of eigenfunctions is developed for a quarterspace with one boundary semi-rigid and the other stress-free. This is the simplest example of a

nonparallel boundary-value problem. A class of solutions is constructed by substituting an appropriate expansion into the wave equation according to a certain "ansatz" about the solution. Out of this class an orthonormal set of functions is constructed and the completeness of this set is verified.

Further, using this complete set of eigenfunctions, an exact expression for the Green's function for this problem is developed. This expression turns out to be a triple integral which can be evaluated analytically. Therefore the response evaluated here is exact.

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1. Introduction

1.1 Hierarchy of elastodynamics problems

The analysis of transient waves in elastic media started in the mid nineteenth century with Stokes' (1849) derivation of an exact solution for a concentrated point force in an unbounded region. Following that, exact solutions for elastic wave propagation problems in a bounded region, for stress-free or bimaterial boundary conditions, remained unsolved for more than half a century due to the difficulties related to conversion of compressional and shear waves into each other and excitation of interface waves at the boundaries. Lamb (1904) opened the door for exact solutions for a class of bounded region problems (boundary surface expressible in terms of one level surface of a coordinate system) by solving the problem of a concentrated point source in a halfspace. More than fifty years later, after the generalized ray theory was introduced, exact solutions were derived for geometries with parallel boundary surfaces i. e. having all boundary surfaces expressible in terms of the same type of level surface of a coordinate system. This was a remarkable accomplishment even though it took more than a century to achieve.

Further evolution in the field of wave propagation in bounded regions was the development of exact solutions for

geometries with nonparallel boundary surfaces. The study of this class of problems started around the turn of the century with Sommerfeld's (1896) investigation of an acoustic wave in a wedge shaped region. However, serious attention to these problems began only in the 1960's.

To describe nonparallel boundary surfaces, more than one level surface of a coordinate system (type A) or, in extreme cases, more than one coordinate system (type B) are required. Here, due to the presence of many different level surfaces, the problem of conversion and excitation of waves at the boundary surfaces is of greater complexity. In addition, the problem of singular behaviour of waves at the edge in type A and transformation of the solution from one coordinate system to another in type B complicates the situation even further. Therefore, the derivation of exact solutions for nonparallel boundary surfaces is a challenging problem for applied mathematicians. Dipping structures, bore-holes, continental margins, mountain roots, ridges, folds, faults and irregular topographies are all expressible only in terms of nonparallel surfaces. Therefore, the solution of these problems is of great interest to seismologists. Furthermore, the solution of these problems is also of interest to engineers for the study of problems such as earthquake resistance of structures and scattering problems during nondestructive testing of materials.

The complex mathematical nature of nonparallel boundary surface problems has made the standard approaches for the

derivation of exact solution ineffective. Therefore, most attempts with these problems have been directed to a very specific part of the elastic field rather than to a general solution for the full elastic field.

In this thesis a procedure for developing an exact solution of the full elastic field for a class of nonparallel boundary surfaces, that requires more than one level surface of a coordinate system, is discussed. The simplest of this class, the quarterspace, is chosen as a specific example for demonstration of the procedure. The approach taken involves finding the complete set of eigenfunctions, for a given geometry and boundary conditions. Out of that complete set of eigenfunctions, Green's functions can be constructed using standard mathematical techniques. For any source in the same geometry and boundary conditions, the full elastic field can be written as a convolution of the appropriate Green's function with the given source function. Thus, in principle, the problem is solved.

1.2 Classification of elastodynamics problems

The elastodynamics problems, which involve boundaries expressible in terms of level surfaces of canonical coordinate systems, can be classified, from the boundary surface point of view, as follows:

1. Type I: Boundary expressible as a level surface of a coordinate system.

A halfspace or a spherical medium in a fullspace are the simplest examples of this class. Examples of higher degree of complexity in this class are the multilayered planar or concentric spherical shell models of the earth. To describe these geometries the same type of level surface of a coordinate system is required several times. With the advent of generalized ray theory, a unified approach to the exact solution of wave propagation problems for this class was made available.

2. Type II: Boundary that requries more than one level surface of a coordinate system.

The simplest examples of this class are wedges and cylindrical holes in halfspaces. Here, intersecting level surfaces form an edge that introduces an additional source of diffracted waves. A multilayered wedge (Figure 1.1) or two multilayered wedges in contact (Figure 1.2) are example of geometries of a higher complexity. Exact solutions for the full elastic field, even for the simplest geometries of this class, are not known to exist.

Figure 1.1 A multilayered wedge

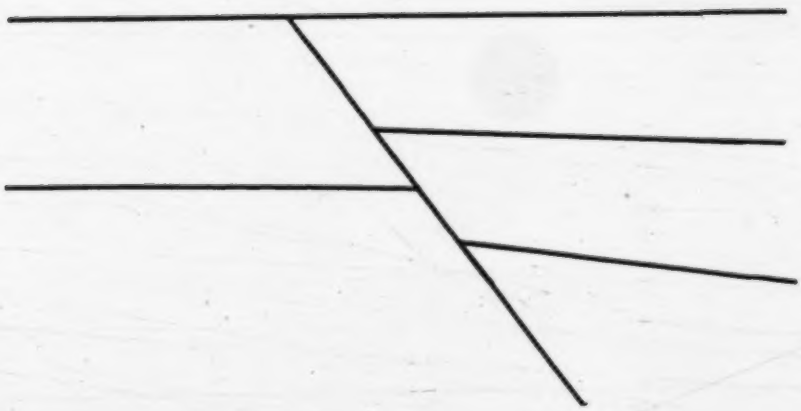

Figure 1.2 Two multilayered wedges in contact

3. Type III: Boundary that requries level surfaces of more than one coordinate system.

A model consisting of a spherical object in a halfspace is a representative of this class. Here, in order to satisfy the boundary conditions, solutions have to be transformed from one coordinate system to another. In this process a transformation matrix of infinite dimension is introduced which has made a computable exact solution unobtainable so far.

1.3 Evolution in methodology for solving type II problems

The method of images was the earliest approach used for theoretical investigations of wave propagation in media bounded by nonparallel surfaces, as applied by Sommerfeld (1896) for scalar waves in a wedge. However, the same approach failed to provide results for elastic waves. In the elastic case, conversion and excitation of waves at the boundary surfaces prevented the construction of images that satisfied the boundary conditions simultaneously at two different surfaces.

A serious look at the nonparallel elastodynamic problem started in the sixties with Knopoff's investigation of wedge type geometries. Such geometries, apart from historical reasons, have direct geophysical applications and are easily adopted for the experimental study of elastic wave

propagation. Therefore, they were the logical first choice for theoretical investigations.

The success gained in solving type I problems with integral transforms suggested that one could use similar techniques for wedge problems. With the integral transform approach the wedge problem was reduced to a problem of solving a set of simultaneous integral equations. It was not possible to carry the analysis any further in a manner similar to type I problems.

In the case of type I problems, the solutions are constructed by pairing the incident (longitudinal or transverse) and the reflected (both longitudinal and transverse) waves, obtained from the equations after imposing the boundary conditions. In the presence of additional mode converting surfaces one is unable to follow the same procedure. However, to gain some insight into this system of simultaneous equations arising in the elastic case, Lapwood (1961) and Kane and Spence (1963) have used an iterative scheme to match the boundary conditions at the surfaces, while Forristall and Ingram (1971) have reduced the system to a singular integral equation to be solved numerically. Thus, for the wedge problems, integral transforms have so far not been successfully applied for deriving solutions in closed form.

The techniques of conical flow and Wiener-Hopf methods, that are quite often used in mechanics problems, suffered a similar fate for the wedge problem. In an excellent review

article by Knopoff (1969), various mathematical techniques that have been applied, with limited success, to wedge problems, are extensively discussed.

Realizing that the multitude of algebraic complications arising in the study of the full elastic field obscures the difficulties associated with the physics of the problem, Knopoff and his collaborators started to focus on a specific aspect of the full field, namely, the surface waves. In addition, the experimental observations due to de Bremaecker (1958), Knopoff and Gangi (1960), Pilant, Knopoff and Schwab (1964) on the transmission around the corner and the reflection of Rayleigh waves by a wedge, made such undertakings more appropriate for a theoretical study.

If the Green's function for a given region is known and the fields of displacements and stresses at the boundary are specified, the displacement field in the interior of that region can be calculated using the representation theorem. For the problem of a surface wave propagating in a wedge, neither the appropriate Green's function nor the displacements and the stresses due to the reflected and transmitted (around the corner) waves at the boundary are known. Therefore, a direct application of this approach is difficult.

Hudson and Knopoff (1964 A, 1964 B) as well as Mal and Knopoff (1966) developed a series of approximations for applying this approach to the problem of Rayleigh waves normally incident upon the corner for an homogeneous elastic

wedge formed by two stress-free planes. These authors used the Green's function for a stress-free halfspace. Such a substitution is valid only if the intensity of the reflected wave is negligible compared to the intensity of the incident wave. Moreover, with this choice of Green's function, virtual stresses are to be applied alternately at each surface (and that results in a infinite iteration loop) to satisfy the stress-free boundary condition. These stresses can be considered to be the sources of secondary fields that radiate Rayleigh waves along with body waves.

Ignoring the body wave parts completely they calculated the transmission and the reflection coefficients. In this calculation they considered that initially three Rayleigh waves were excited. The first one was a reflected (outgoing) wave along the surface parallel (Sp) to the incident wave. The other two were, one going towards the corner (incoming) and a second one going away from the corner (outgoing), along the surface oblique (Sq) to the incident wave. This incoming wave, generated along Sq, in turn excited an outgoing wave along Sq and incoming and outgoing waves along Sp. This cycle was repeated indefinitely. The theoretical values of the transmission and the reflection coefficients thus obtained, agreed sufficiently well with the experimental observations of Pilant, Knopoff and Schwab (1964) for obtuse wedge angles. Taking into account the interaction of the outgoing Rayleigh wave with the wedge surfaces, Viswanathan, Kuo and Lapwood (1971) obtained

higher order approximations that gave considerably closer agreement with the experimental data.

However, significant disagreement with the experimental data still existed for acute wedge angles. This was due to the fact that waves of a particular kind generated in one iterative cycle excited waves of the other kinds in the next stage of iteration. Therefore, by neglecting the body wave parts one essentially dropped the contributions due to body wave induced excitations of the Rayleigh wave and vice-versa. As the contributions from such interactions became dominant for acute wedge angles, these approximate theories broke down in that region. Although Lopez (1967) tried to incorporate contributions from body wave terms the results did not improve appreciably due to algebraic complications.

The described papers illustrate the fact that one can not build an exact theory for elastic wave propagation in a wedge, since an accurate calculation of the transmission and the reflection coefficients is still not feasible. So far, in a wedge geometry it was only possible to construct an exact theory for SH wave propagation. However, this is a scalar wave problem and we are actually interested in vector wave propagation due to interacting P-SV waves. Moreover, the theories developed for SH waves in a wedge can not be applied directly for SH waves in a multilayered wedge: for example to a wedge representing a halfspace with a dipping structure. An approximate solution for dipping problems was

developed by Hong and Helmberger (1977). They expanded the solution for SH wave propagation in a wedge with one surface rigid and another stress-free, as derived by Hudson (1963), into ray integrals and evaluated them using the Cagniard-de Hoop method. They then extended these solutions to the dipping case by just changing the reflection coefficients at the rigid boundary to those of an elastic halfspace. Pao and Zeigler (1982) reformulated this problem in terms of generalized ray integrals. They expressed the SH-source ray integral in two systems of coordinates, one for each of the two nonparallel surfaces, and constructed the ray integrals for multi-reflected waves within the wedge. The obtained transient SH wave solution was exact up to the time of the arrival of the diffacted waves from the edge. This approach was extended to SH waves in several dipping layers by Zeigler and Pao (1985). In all these approaches described above, the diffraction effect due to the edge was not taken into account.

The case of two multilayered wedges in welded contact provides a reasonable model for a variety of problems such as a transition zone between lithospheric plates, horizontal or dipping layers with a fault, heterogeneous structures made up of fine homogeneous layers, etc. Furthermore, field observations of reflected surface waves from deep faults (Sikharulidze, 1969) and Rayleigh waves transmitted across a continental margin (Capon, 1970) are reported. In addition, the study of seismic behavior in the above mentioned models

can also provide insight into the problems of diffraction from sharp corners. Therefore, the theoretical investigation of such models is of considerable interest not only for observational seismology but also for exploration seismology. For this reason there have been many attempts to construct a theory, even an approximate one, for seismic waves in such models.

Alsop and his co-workers considered a simple geometry of the above mentioned class, namely two multilayered quarterspaces in welded contact. Furthermore, they restricted themselves to calculating the transmission and the reflection coefficients for a normal incident Love wave (Alsop, 1966) and a similar Rayleigh wave (McGarr and Alsop, 1967), incident upon the plane of vertical discontinuity. Because for such situations the intensity of the body wave part is negligible, they assumed that the scattered field consisted of waves of incident type only. They expanded the scattered field in terms of a complete set of eigenfunctions, of the incident wave type, derived from the corresponding multilayered halfspace problem. With such basis functions the boundary conditions on the horizontal interfaces were automatically satisfied. However, the boundary conditions on the vertical interface were not satisfied exactly, because the body wave part was not taken into account in the scattered field. Hence, those boundary conditions were satisfied only approximately. For this reason, they used a variational method determining the

coefficients of expansion of the scattered field by minimizing the mean square matching error for the displacements and stresses at the vertical interface. The experimental data due to McGarr and Alsop (1967) showed some degree of agreement with the computed results. One reason for not obtaining a better agreement was the fact that in many cases the basic assumption that the body wave energy was negligible in the neighbourhood of the boundary, was not satisfied.

In order to improve the agreement with experimental results Alsop and his collaborators (Alsop, Goodman and Gregersen, 1974; Gregresen and Alsop, 1974, 1976; Gregersen, 1978; Chen and Alsop, 1979) developed another approximate theory. Here the incident as well as the reflected and transmitted surface waves were represented as superpositions of homogeneous and inhomogeneous body waves. The solution of the problem of propagation of such waves across a vertical interface was determined from the assumption that these homogeneous and inhomogeneous waves together satisfy the boundary conditions at the vertical interface, but do not satisfy the boundary conditions at the horizontal interfaces. The values of the transmission and reflection coefficients thus obtained agreed better with the experimental data. However, by not taking into account the boundary conditions at the horizontal interfaces, diffraction effects were ignored in this method.

Theoretical studies on the transmission and reflection of surface waves in the above mentioned model were also presented by Ets and Yanovskaya (for an extended biliography see the review article by Ets and Yanovskaya, 1985) using representation theorems. In their calculations they assumed that the Green's function in each quarterspace is the same as in the corresponding halfspace. Because such a Green's function can not account for the wave field reflected from the vertical interface, they added a few terms to the Green's function to make the necessary correction. This approximate theory gave a suitable result as long as the effect of the body wave field could be ignored.

Numerical approaches have also been carried out for some type II problems. Alterman and Rotenberg (1969) treated successfully the case of a quarterspace with stress-free surfaces excited by a buried source, using a finite difference scheme. Lysmer and Drake (1971) developped a finite element technique to study the propagation of Love waves across nonhorizontal layered structures. Drake (1972A, 1972B) extended that approach to Rayleigh waves across continental margins.

Very recently there has been some progress in using analytical methods to study the elastic field for the P-SV case. Miklowicz (1982) has derived an expression for the transient response due to a pressure step applied to the surface of a quaterspace, which has zero shear stress along that surface and is traction free along the other surface.

Here a one-sided Laplace transform which requires a priori information about the displacements and their gradients at the boundaries was used. Zeigler and Pao (1984) extended their analysis of SH waves in a multilayered wedge to the P-SV case. This approach did not take surface waves into account.

One of the most elegant but least applied techniques of elastodynamics is the expansion of a wave field in a series of eigenfunctions. This method represents an extension of the Sturm-Liouville theory of ordinary differential equations. Here, in order to solve transient problems, one has to construct a complete set of eigenfunctions that satisfies the homogeneous elastic wave equation and the given set of boundary conditions.

In this thesis, an eigenfunction expansion approach is taken for the type II problems, although in many problems applicability of this method is restricted by the difficulties related with the construction of the eigenfunctions. For the demonstration of this procedure, we consider a quarterspace geometry with one boundary semi-rigid and the other stress-free. In chapter 2 the problem is formulated for this quarterspace geometry. In chapter 3 the eigenfuntions for such a geometry are developed. By substituting an appropriate "ansatz" into the wave equation, a class of solutions is constructed. Out of that class an orthonormal set of functions is formed after which completeness of that set is verified. In chapter 4, an

expression for the displacement field in this geometry due to a line source is developed in terms of the obtained eigenfunctions and the numerical evaluation of that expression is carried out. In the expression for the displacement field it is possible to identify terms similar to the PP, PS, SS and SP displacement field terms of Lamb's problem. It is an accepted fact in the literature (Aki and Richards 1980) that these terms diverge for large times individually but on addition cancel their singular behaviour of each other so that the resultant displacement field is a well behaved function. In our approach, the displacement field is separated into terms that are well behaved and no cancellations takes palce. This is an additional result discovered while investigating the present problem. In chapter 5 a discussion of possible future directions in elastodynamics problems in media bounded by nonparallel surfaces is presented.

2. Elastodynamics of a quarterspace geometry

2.1 Choice of the boundary conditions

A quarterspace ,i.e., a quadrant of a full space, is the simplest example of a nonparallel geometry. For this geometry the boundaries can be expressed, in a Cartesian coordinate system as two different level surfaces, which are perpendicular to each other. On the boundary surfaces one has to consider proper boundary conditions. For the eigenvalue problem to be well-posed mathematically, the wave equation plus boundary conditions must be self-adjoint. This leads to the three possible boundary conditions which are enumerated below.

The simplest boundary condition one may choose on a surface is semi-rigid. On such a surface normal component of the displacement field and tangential components of the stress vanish. The other possible boundary conditions that can be considered are rigid and stress-free. The rigid boundary condition has both the normal and tangential components of the displacement field vanishing whereas the stress-free condition has both the normal and tangential components of the stress vanishing.

Thus, there are the following six possible choices of boundary conditions for a quarterspace:

1. Both surfaces are semi-rigid,
2. One surface is semi-rigid and the other is rigid,

3. Both surfaces are rigid,
4. One surface is semi-rigid and the other is stress-free,
5. One surface is rigid and the other is stress-free,
6. Both surfaces are stress-free.

The non-separability in nonparallel geometries comes from mode conversion (longitudinal to transverse and vice-versa) and the excitation of evanescent or interface waves at the boundaries. It is a known fact that semi-rigid surfaces reflect elastic fields without mode conversion and excitation of evanescent waves. On the other hand rigid surfaces reflect elastic fields with only mode conversion whereas stress-free surfaces produce mode conversion as well as the excitation of evanescent waves. Thus, the first choice, a quarterspace with both surfaces semi-rigid is a separable problem and the solution is standard and well known. Actually this type of elastodynamic problem is equivalent to an acoustic problem for which solutions have been known since as early as 1896 (Sommerfeld). The second and third choices of the boundary conditions involve mode conversion at one and both boundaries respectively. Here, the solution for the elastodynamic problem can also be obtained without much difficulty.

The remaining choices of boundary conditions lead to non-separable problems. However, the fourth choice leads to a simpler problem because it involves mode conversion and the excitation of evanescent waves at only one surface and

pure reflection at the other surface. In the hierarchy of complexity the fifth choice comes next. It involves mode conversion at both surfaces and the excitation of evanescent waves at only one of the surfaces. The most difficult boundary condition problem to solve is the sixth choice which involves mode conversion as well as the excitation of evanescent waves at both surfaces. For simplicity, we study, the elastodynamic problem for a quarterspace with one surface semi-rigid and the other stress-free.

Moreover, apart from simplicity of the mathematical analysis, this choice of boundary condition also provides a realistic model for some geological structures such as, for example, a vertical contact of metamorphic and sedimentary rock.

The particle vibrations along the direction parallel to the metamorphic and sedimentary rock contact is easier than along the perpendicular to the contact, i.e. towards the metamorphic rock. For waves incident from the sedimentary side of the contact, towards the metamorphic regime, the particle movements, could sometimes cease to exist completely in the perpendicular direction. In this case the contact could be regarded as semi-rigid.

2.2 Formulation

Let $\vec{u}(\vec{x},t)$ be the displacement field in an homogeneous isotropic quarterspace defined by $0 \leq x_1 \leq \infty$ and $0 \leq x_2 \leq \infty$,

$-\infty \leq x_3 \leq \infty$ (Figure 2.1), obeying the elastic wave equation

$$C_p^2 \nabla(\nabla\cdot\vec{u}) - C_s^2 \nabla\times\nabla\times\vec{u} = \frac{\partial^2\vec{u}}{\partial t^2} \quad . \tag{2.1}$$

Here C_p and C_s are the P- and S-wave velocities, related to the Lamé constants μ and λ and density ρ by:

$$C_p^2 = \frac{\lambda+2\mu}{\rho} \quad , \quad C_s^2 = \frac{\mu}{\rho} \quad .$$

The displacement field $\vec{u}(\vec{x},t)$ is subject to the following boundary conditions:
the normal component of the displacement field and tangential components of the stress vanish at the surface $x_1 = 0$

$$u_1 = 0 \quad \text{at} \quad x_1 = 0, \tag{2.2}$$

$$\sigma_{12} = \sigma_{13} = 0 \quad \text{at} \quad x_1 = 0,$$

and both the normal and tangential components of the stress vanish at the surface $x_2 = 0$

$$\sigma_{21} = \sigma_{22} = \sigma_{23} = 0 \quad \text{at } x_2 = 0. \tag{2.3}$$

Assuming harmonic time dependence

$$\vec{u}(\vec{x},t) = e^{-i\omega t}\,\vec{u}(\vec{x}) \tag{2.4}$$

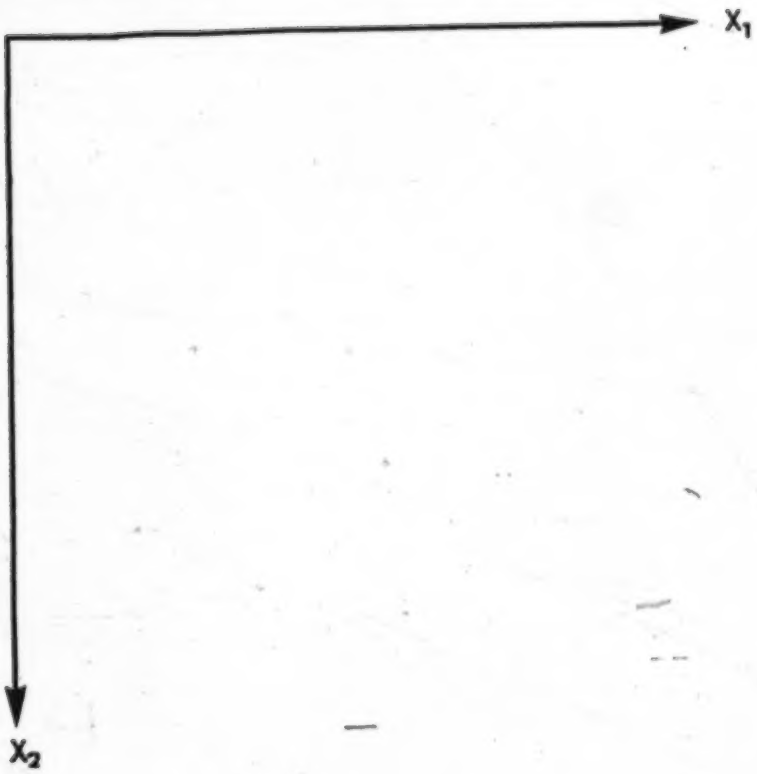

Figure 2.1 A quarterspace geometry and coordinate system

and introducing the notation

$$\vec{\mathcal{D}} \equiv C_p^2 \nabla(\nabla\cdot\ ) - C_s^2 \nabla\times\nabla\times \tag{2.5}$$

the wave equation (2.1) is reduced to an eigenvalue problem

$$\vec{\mathcal{D}}\ \vec{u}(\vec{x}) = -\omega^2\ \vec{u}(\vec{x})\ . \tag{2.6}$$

The operator $\vec{\mathcal{D}}$ is self-adjoint for the given boundary conditions. For a self-adjoint operator a complete set of eigenfunctions always exists. Therefore, the problem here is reduced to finding all the eigenfunctions, $\vec{u}^{(m)}(\vec{x})$, of equation (2.6). The superscript "m" stands for a set of parameters needed to completely label an eigenfunction.

The orthogonality of these eigenfunctions is expressed as

$$<\vec{u}^{(m)}(\vec{x}),\ \vec{u}^{(m')}(\vec{x})> = 0 \quad \text{for} \quad \omega^{(m)} \neq \omega^{(m')}\ . \tag{2.7}$$

Here $\omega^{(m)}$, $\omega^{(m')}$ are the corresponding eigenvalues and the bracket $<,>$ denotes the scalar product, defined by:

$$<\vec{f},\ \vec{g}> = \int_0^\infty dx_1 \int_0^\infty dx_2\ f_i^*\ g_i \tag{2.8}$$

where * stands for complex conjugate and the Einstein summation convention is used.

Furthermore, these normalized eigenfunctions satisfy the following closure property:

$$\sum_{m} \int u_i^{(m)}(\vec{x})\, u_j^{(m)}(\vec{x}') = \delta_{ij}\, \delta(\vec{x}-\vec{x}') \quad . \tag{2.9}$$

The superposed summation and integration signs over the index "m" mean summation over all discrete parameters and integration over the continuous parameters.

2.3 Decomposition of the Wave Equation

The equation (2.6) constitutes a system of three simultaneous partial differential equations for the three components of the displacement field. The first task at hand is to find solutions to this system of equations that simultaneously obey the boundary conditions (2.2)-(2.3). Once such solutions are obtained, we normalize them in order to have a complete set of "eigensolutions", i.e. satisfying the orthonormality condition (2.8) and the completeness relation (2.9).

For two-dimensional and three-dimensional problems one may reduce the vector equation (2.6) to, respectively a fourth or sixth order partial differential equation for each component of the displacement field, except the decoupled component of the displacement field in the two-dimensional case, which satisfies a second order equation. However, to develop solutions for such higher order partial differential

equations with given boundary conditions is a formidable task.

Another approach would be to try to reduce this system of partial differential equations to a system of coupled ordinary differential equations or albegraic ones, by using an integral transform approach, e.g. the Laplace transform. Due to the semi-infinite nature of the spatial coordinates in this geometry, the knowledge of the transformed quantities at zero coordinates is required in order to use such an integral transform method. Because the complete set of eigenfunctions of a given system of differential equations involves the free solutions of that system, we may not be able to choose only a subset. On the other hand, by choosing the displacement field at the surface we might, however, restrict ourselves to a too small subset of acceptable solutions. Thus, this approach is also of no use in our problem.

It turns out that the decomposition of equation (2.6) in terms of the Lamé potentials is a most suitable. In this approach a vector field is decomposed into a curl-free part and a divergence-free part. With such a choice, the vector equation (2.6) is reduced to a set of three decoupled scalar Helmholtz equations. The scalar potentials satisfying these equations are, however, still coupled through the boundary conditions. Nevertheless, it is possible to construct proper solutions satisfying all the conditions, by using the wide knowledge about solving the Helmholtz equation.

Having a line source running parallel to the x_3-axis, one may choose a two-dimensional geometry. For such a choice, the displacement field in terms of the potentials is defined as:

$$\vec{u} = \nabla\phi + \nabla\times\vec{I}_3\psi + \vec{I}_3 u_3 \ . \tag{2.10}$$

This reduces equation (2.6) to the following set of decoupled scalar equations:

$$(\nabla^2 + K_p^2)\ \phi(\vec{x},\omega) = 0 \ , \tag{2.11}$$

$$(\nabla^2 + K_s^2)\ \psi(\vec{x},\omega) = 0 \ , \tag{2.12}$$

$$(\nabla^2 + K_s^2)\ u_3(\vec{x},\omega) = 0 \ , \tag{2.13}$$

where

$$K_p = \frac{\omega}{C_p}, \quad K_s = \frac{\omega}{C_s}, \quad \nabla^2 = \partial_1^2 + \partial_2^2 \ . \tag{2.14}$$

The boundary conditions take the form:

$$u_1 = \partial_1\phi + \partial_2\psi = 0 \qquad \text{at } x_1 = 0, \tag{2.15}$$

$$\sigma_{12} = \mu\{(2\partial_1\partial_2\phi + (\partial_2^2 - \partial_1^2)\psi\} = 0 \qquad \text{at } x_1 = 0, \tag{2.16}$$

$$\sigma_{21} = \mu[(2\partial_1\partial_2\phi + (\partial_2^2-\partial_1^2)\psi] = 0 \qquad \text{at } x_2 = 0, \qquad (2.17)$$

$$\sigma_{22} = \lambda\nabla^2\phi + 2\mu[\partial_2^2\phi - \partial_2\partial_1\psi] = 0 \qquad \text{at } x_2 = 0, \qquad (2.18)$$

$$\sigma_{13} = \mu\partial_1 u_3 = 0 \qquad \text{at } x_1 = 0, \qquad (2.19)$$

$$\sigma_{23} = \mu\partial_2 u_3 = 0 \qquad \text{at } x_2 = 0. \qquad (2.20)$$

Here one observes that the u_3 component of the displacement field is completely decoupled from ϕ and ψ; it obeys the differential equation (2.13) and the boundary conditions (2.19)-(2.20) above. Therefore, one may construct a class of solutions for the differential equation (2.6) out of u_3 only, by choosing $\phi = \psi = 0$. Another class of solutions can be constructed out of ϕ and ψ only, setting $u_3 = 0$. Furthermore, although ϕ and ψ obey separate differential equations, they are still coupled through the boundary conditions (2.15)-(2.18). Choosing either of them equal to zero and satisfying the boundary conditions with only one potential, one gets only the trivial solution for that potential. This implies that both potentials have to exist together.

In the usual terminology of seismology the former class of solutions corresponds to the SH case while the latter corresponds to the P-SV case.

2.4 The Solution

The following four classes of solutions for a two-dimensional scalar Helmholtz equation can be constructed under the "ansatz" that the solution is a product of exponential functions of two spatial coordinates:

1. Both exponential functions are harmonic,
2. Exponential function of x_1 is harmonic and of x_2 is damped,
3. Exponential function of x_1 is damped and of x_2 is harmonic,
4. Both exponential functions are damped.

In the process of the excitation of interface waves at the boundary surfaces, damped waves arise. To describe such waves, damped solutions of the Helmholtz equation are required. In the problem under consideration, the surface $x_1=0$ does not allow the excitation of interface waves, so there are no solutions having a damped exponential function with x_1 in the argument. Thus, only the first two types of solutions of the Helmholtz equation are required.

The general solution of a scalar Helmholtz equation (e.g. equation (2.11)) containing only harmonic waves of type 1 can be written in terms of a Fourier integral as

$$\phi(\vec{x},\omega) = \int_{-\infty}^{+\infty} d\ell \int_{-\infty}^{+\infty} dk\, H(k,\ell)\, e^{ikx_1+i\ell x_2}\, \delta(k^2+\ell^2-K_p^2)\ . \tag{2.21}$$

Here the Fourier coefficient $H(k,\ell)$ is defined with a delta function to ensure that the Helmholtz equation is satisfied automatically. Using the identity

$$\delta(k^2+\ell^2-K_p^2) = \frac{1}{2\sqrt{K_p^2-k^2}}\left[\delta(\ell-\sqrt{K_p^2-k^2}) + \delta(\ell+\sqrt{K_p^2-k^2})\right]\Theta(K_p^2-k^2), \quad (2.22)$$

where Θ is the Heaviside function and performing the integration with respect to ℓ followed by, for the negative values of k, the change of variable $k \longrightarrow -k$ and reexpressing the exponentials that occur in the integrand, in terms of trigonometric functions, we have:

$$\phi(\vec{x},\omega) = \int_0^{K_p} dk\ \{H_1(k,\alpha)\ \mathrm{Cos}kx_1\ \mathrm{Cos}\alpha x_2 + H_2(k,\alpha)\ \mathrm{Cos}kx_1\ \mathrm{Sin}\alpha x_2 + H_3(k,\alpha)\ \mathrm{Sin}kx_1\ \mathrm{Cos}\alpha x_2 + H_4(k,\alpha)\ \mathrm{Sin}kx_1\ \mathrm{Sin}\alpha x_2\}\ . \quad (2.23)$$

Here we have introduced $\sqrt{K_p^2-k^2}=\alpha$. In the process carrying out ℓ integration and the change of variable for the negative values of k, we endup with Fourier coefficients that have four different arguments. H_j, j=1,4 are the linear combinations of these coefficients.

Later on it will become apparent that the above choice of representation for the solution is quite helpful because

the geometries of interest here are semi-infinite, and both sine and cosine functions form a complete set over such a range. That is,

$$\frac{2}{\pi}\int_0^\infty dx\ \mathrm{Cos}kx\ \mathrm{Cos}k'x = \frac{2}{\pi}\int_0^\infty dx\ \mathrm{sin}kx\ \mathrm{sin}k'x = \delta(k-k')\ . \tag{2.24}$$

A general solution containing only damped waves of type 2 for $\phi(\vec{x},\omega)$ can be written as

$$\phi(\vec{x},\omega) = \int_0^\infty d\ell \int_0^\infty dk\ D(k,\ell)\ e^{ikx_1 - \ell x_2}\ \delta(k^2-\ell^2-K_p^2)\ . \tag{2.25}$$

Here $D(k,\ell)$ is the Fourier coefficient. As before, integrating with respect to ℓ, change of the variable for the negative values of k, rearranging the terms and redefining the coefficients we have:

$$\phi(\vec{x},\omega) = \int_{K_p}^\infty dk \left[D_1(k,\Gamma)\ \mathrm{Cos}kx_1\ e^{-\Gamma x_2} + D_2(k,\Gamma)\ \mathrm{Sin}kx_1 e^{-\Gamma x_2} \right]\ . \tag{2.26}$$

Here $\Gamma = \sqrt{k^2 - K_p^2}$. Actually (2.26) is an analytic continuation of (2.23) for the range $k \geq K_p$.

Similarly the general solutions, for $\psi(\vec{x},\omega)$ and $u_3(\vec{x},\omega)$ are obtained by replacing α by $\beta(=\sqrt{K_s^2 - k^2})$ in equation (2.23), Γ by $\Delta(=\sqrt{k^2 - K_s^2})$ in equation (2.26) and changing the limit of integration from K_p to K_s respectively.

Thus, each of the potentials is expressed in terms of four harmonic solutions and two damped solutions. With such elementary solutions one can construct the eigenfunctions to represent the solution of the given problem.

As pointed out in the previous section, the solution of the SH case is determined by u_3 alone. This is a straightforward task. On the other hand, for the P-SV case both of the potentials ϕ and ψ are required. Now ϕ and ψ have different threshholds for the transition from harmonic to damped behaviour, K_p and K_s $(K_p<K_s)$, respectively; and as mentioned earlier, both ϕ and ψ have to co-exist for non-trivial solutions. Therefore, for the P-SV problem one has three classes of the ϕ and ψ pair:

1. Harmonic

   k ranges from 0 to K_p and both ϕ and ψ are harmonic.

2. Damped type I

   Here k ranges from K_p to K_s and ϕ is damped while ψ is harmonic.

3. Damped type II

   k ranges from K_s to ∞ and both ϕ and ψ are damped.

In every class of solutions we considered all possible combinations of elementary solutions, in order to satisfy the boundary conditions. Each of these linearly independent combinations leading to a non-trivial solution of the

quarterspace is called an "eigenmode".

We found that three parameters are needed to distinguish an eigenfunction: a discrete index n to label the eigenmode and two continuous variables, denoted by k and ω. One may of course also choose the components of the propagation vector, k_1 and k_2, but the choice $k_1 = k$ and ω, where $k_1^2 + k_2^2 = \frac{\omega^2}{C^2}$, C being the apparent velocity along the surface, turns out to be very convenient. Thus

$$m = (n,k,\omega) \ . \tag{2.27}$$

While labelling eigenfunctions we shall write n as a superscript and k and ω as arguments.

3. Eigenfunctions

3.1 The construction of Eigenmodes

The geometry being two-dimensional, the u_3 component of the displacement field determines solely the SH wave propagation while the Lamé potentials, ϕ and ψ, govern together the P-SV waves propagation. Thus, the SH and P-SV waves propagations are independent of each other. Although the vector wave propagation connected with the P-SV waves is the only quantity of real interest, the eigenfunctions for the scalar wave propagation arising due to SH waves are also presented here, for the sake of completeness.

Each of ϕ, ψ and u_3 has four oscillatory and two non-oscillatory elementary solutions. Various linear combinations of these elementary solutions are considered as an "ansatz" for satisfying the appropriate boundary conditions. At first, we only take one elementary solution at a time in the "ansatz". If we obtain only trivial solutions then we take a linear combination of two elementary solutions. If these also fail then we try a linear combination of three elementary solutions and so on. In this way we generate a set of solutions. Now we normalize these solutions so that they satisfy the orthonormality relation (2.8). To be able do so, in general, one may require the use of the Gram-Schmidt othonormalization procedure. In our case, the solutions constructed from

elementary solutions, are by themself mutually orthogonal. Thus, no such orthogonalization procedure is required. Later on we verify completeness of this set to ensure that all possible eigenfunctions have been found out.

3.1.1 Harmonic modes

In the harmonic region ($0 \leq k \leq K_p$) each of the potentials ϕ and ψ has four types of elementary solutions. Taking only one elementary solution at a time as a "ansatz", we find sixteen types of combinations for the ϕ and ψ pair. Working through the algebra one finds that only two of them are non-trivial. The first of these is:

$$\begin{pmatrix} \phi^{(1)} \\ \\ \psi^{(1)} \end{pmatrix} = \begin{pmatrix} a_1(k,\alpha)\ \mathrm{Cos}kx_1\ \mathrm{Cos}\alpha x_2 \\ \\ a_2(k,\beta)\ \mathrm{Sin}kx_1\ \mathrm{Sin}\beta x_2 \end{pmatrix} \Theta(K_p-k) \quad . \quad (3.1)$$

Here Θ is the Heaviside function and a_1 and a_2 are unknown coefficients to be fixed by the boundary conditions (2.15)-(2.18). This pair of elementary solutions automatically satisfies the boundary conditions (2.15)-(2.17). Equation (2.18) yields:

$$\sigma_{zz}\Big|_{x_2=0} = \{(\beta^2-k^2)\ a_1 + 2k\beta a_2\}\ \mathrm{cos}kx_1 = 0, \quad (3.2)$$

which requires that

$$a_2 = - a_1 \frac{(\beta^2 - k^2)}{2k\beta} . \qquad (3.3)$$

Then one has

$$\begin{pmatrix} \phi^{(1)} \\ \psi^{(1)} \end{pmatrix} = a_1 \begin{pmatrix} \text{Cos}kx_1 \, \text{Cos}\alpha x_2 \\ - \frac{(\beta^2 - k^2)}{2 k \beta} \text{Sin}kx_1 \, \text{Sin}\beta x_2 \end{pmatrix} \Theta(K_p - k). \qquad (3.4)$$

The constant a_1 is fixed by computing the displacement field, as defined by equation (2.10) and normalizing in the sense of the relation (2.8). This yields the following eigenmode:

$$\vec{u}^{(1)}(\vec{x}, k, \omega) = \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{bmatrix} = - \frac{2}{\pi} \frac{2k\sqrt{\beta/\omega}}{\sqrt{(\beta^2 - k^2)^2 + 4k^2\alpha\beta}} \times$$

$$\begin{bmatrix} \text{Sin}kx_1 \{k\text{Cos}\alpha x_2 + \frac{(\beta^2 - k^2)}{2 k} \text{Cos}\beta x_2\} \\ \text{Cos}kx_1 \{\alpha\text{Sin}\alpha x_2 - \frac{(\beta^2 - k^2)}{2 \beta} \text{Sin}\beta x_2\} \\ 0 \end{bmatrix} \Theta(K_p - k). \qquad (3.5)$$

A second set of non-trivial elementary solutions in the harmonic region is:

$$\begin{pmatrix} \phi^{(2)} \\ \psi^{(2)} \end{pmatrix} = \begin{pmatrix} a_3(k,\alpha)\ \mathrm{Cos}kx_1\ \mathrm{Sin}\alpha x_2 \\ a_4(k,\beta)\ \mathrm{Sin}kx_1\ \mathrm{Cos}\beta x_2 \end{pmatrix} \Theta(K_p - k) \qquad (3.6)$$

Going through the analysis as before yields the following eigenmode:

$$\vec{u}^{(2)}(\vec{x},k,\omega) = -\frac{2}{\pi}\frac{(\beta^2 - k^2)}{\sqrt{(\beta^2-k^2)^2 + 4k^2\alpha\beta}}\frac{1}{\sqrt{\omega\alpha}} \times$$

$$\begin{bmatrix} \mathrm{Sin}kx_1\{k\mathrm{Sin}\alpha x_2 - \frac{2k\alpha\beta}{(\beta^2-k^2)}\mathrm{Sin}\beta x_2\} \\ \mathrm{Cos}kx_1\{-\alpha\mathrm{Cos}\alpha x_2 - \frac{2k^2\alpha}{(\beta^2-k^2)}\mathrm{Cos}\beta x_2\} \\ 0 \end{bmatrix} \Theta(K_p - k)\ . \qquad (3.7)$$

These two eigenmodes may be regarded as the P-SV waves in the usual seismological terminology.

3.1.2 Damped modes of type I

In the region ($K_p \leq k \leq K_s$) the potentials ϕ and ψ have two damped and four harmonic elementary solutions repectively. Choosing one elementary solution at a time for

the corresponding potentials yields eight possible permutations for "ansatz". All of them are trivial. Taking linear combinations of two elementary solutions for each potential, one finds that only the following combination is non-trivial:

$$\begin{pmatrix} \phi^{(3)} \\ \psi^{(3)} \end{pmatrix} = \begin{pmatrix} a_5(k,\Gamma)\ \mathrm{Cos}kx,\ e^{-\Gamma x_2} \\ \mathrm{Sin}kx,\ \{a_6(k,\beta)\ \mathrm{Cos}\beta x_2 + a_7(k,\beta)\ \mathrm{Sin}\beta x_2\} \end{pmatrix}$$

$$\Theta(k-K_p)\ \Theta(K_s-k)\ . \qquad (3.8)$$

The boundary conditions (2.15) and (2.16) are satisfied identically, while equations (2.17) and (2.18) give:

$$\sigma_{21}|_{x_2=0} = \{(\beta^2 - k^2)\ a_5 + 2k\beta a_7\}\ \mathrm{Cos}kx, = 0, \qquad (3.9)$$

$$\sigma_{22}|_{x_2=0} = \{2k\Gamma a_5 - (\beta^2 - k^2) a_6\}\ \mathrm{Sin}kx, = 0. \qquad (3.10)$$

Thus,

$$a_6 = \frac{2\ k\ \Gamma}{(\beta^2 - k^2)}\ a_5, \qquad (3.11)$$

$$a_7 = -\frac{(\beta^2 - k^2)}{2\ k\ \beta}\ a_5. \qquad (3.12)$$

Fixing a_5 by normalization we find the following eigenmode:

$$\vec{U}^{(3)}(\vec{x},k,\omega) = -\frac{2}{\pi}\frac{2k(\beta^2-k^2)\sqrt{\beta/\omega}}{\sqrt{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}} \times$$

$$\begin{bmatrix} \mathrm{Sin}kx_1\left[ke^{-\Gamma x_2}+\frac{2k\Gamma\beta}{(\beta^2-k^2)}\mathrm{Sin}\beta x_2+\frac{(\beta^2-k^2)}{2k}\mathrm{Cos}\beta x_2\right] \\ \mathrm{Cos}kx_1\left[\Gamma e^{-\Gamma x_2}+\frac{2k^2\Gamma}{(\beta^2-k^2)}\mathrm{Cos}\beta x_2-\frac{(\beta^2-k^2)}{2\beta}\mathrm{Sin}\beta x_2\right] \\ 0 \end{bmatrix} \times$$

$$\Theta(k-K_p)\,\Theta(K_s-k)\ . \quad (3.13)$$

This mode represents an SV-wave reflection at the critical angle. Details of the normalization are presented in Appendix I. This calculation is of interest because, despite the presence of the non-oscillatory terms, one ends up with delta functions in the normalization relation.

3.1.3 Damped modes of type II

Both potentials have two damped elementary solutions in the region ($K_s \le k \le \infty$). So there are four possible combinations of the elementary solutions. Of these, only the following pair of potentials yields a non-trivial solution:

$$\begin{pmatrix}\phi^{(4)}\\ \psi^{(4)}\end{pmatrix} = \begin{pmatrix} a_s(k,\Gamma)\ \mathrm{Cos}kx_1\ e^{-\Gamma x_2} \\ a_s(k,\Delta)\ \mathrm{Sin}kx_1\ e^{-\Delta x_2}\end{pmatrix}\ \Theta(k-K_s)\ . \quad (3.14)$$

Applying the boundary conditions we find that equations (2.15) and (2.16) are satisfied automatically. The remaining equations (2.17) and (2.18) yield following pair of equations to be satisfied simultaneously:

$$\sigma_{21}|_{x_2=0} = \{2k\Gamma a_1 + (\Delta^2 + k^2)a_3\} \operatorname{Sin} kx_1 = 0, \qquad (3.15)$$

$$\sigma_{22}|_{x_2=0} = \{(\Delta^2 + k^2)a_1 + 2k\Delta a_3\} \operatorname{Cos} kx_1 = 0. \qquad (3.16)$$

This requires that for $k \geq K_s$

$$\begin{bmatrix} 2k\Gamma & (\Delta_2 + k_2) \\ (\Delta_2 + k_2) & 2k\Delta \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} = 0 \qquad (3.17)$$

A non-trivial solution can only exist if the determinant of this matrix vanishes, i.e.,

$$(\Delta^2 + k^2)^2 = 4k^2\Gamma\Delta \qquad (3.18)$$

for all values of k between K_s and ∞. This is the well known secular equation for the Rayleigh wave. It is known that for $\frac{C_p}{C_s} < \sqrt{3}$ it has two real and two complex roots whereas for $\frac{C_p}{C_s} \geq \sqrt{3}$ there are four real roots. In both situations one of the real roots which lies at $k = \infty$ yields a

displacement field zero everywhere. Furtheron, in the former case for the complex roots and in the latter case for two of the real roots the displacement fields do not tend to zero at infinity. Actually these root do not lie on the proper Riemann sheet therefore they are not roots of equation (3.18), hence, they are giving non physical solution. Thus, for both cases, the only admissible root is the real root at

$$k = K_r \qquad \text{where} \qquad K_r = \frac{\omega}{C_r}.$$

C_r is the Rayleigh wave velocity and

$$C_r < C_s.$$

Corresponding to this value of k ($=K_r$) one has the following eigenmode obtained after proper normalization:

$$\vec{u}^{(4)}(\vec{x},k,\omega) = -\sqrt{\frac{1}{\pi}}\,\frac{(\Delta^2 - k^2)\sqrt{\Delta/C_r}}{\sqrt{R(k)}} \times$$

$$\begin{bmatrix} \text{Sinkx}_1 \left[k\, e^{-\Gamma x_2} - \frac{2\,k\,\Gamma\,\Delta}{(\Delta^2+k^2)}\, e^{-\Delta x_2}\right] \\ \text{Coskx}_1 \left[\Gamma\, e^{-\Gamma x_2} - \frac{2\,k^2\,\Delta}{(\Delta^2+k^2)}\, e^{-\Delta x_2}\right] \\ 0 \end{bmatrix} \delta(k - K_r). \qquad (3.19)$$

where

$$R(k) = k^2(\Gamma^2 + k^2)\Delta^2 + k^2(\Delta^2 + k^2)\Gamma^2 - 2k^2\Gamma\Delta(\Delta^2 + \Gamma^2). \tag{3.20}$$

It is evident that this mode is the Rayleigh wave.

3.1.4 SH modes

Setting $\phi = \psi = 0$ and $u_3 \neq 0$ we find a mode which has a displacement field parallel to the boundary or a SH mode. For the harmonic region there are four elementary solutions, but only one, namely, $\mathrm{Cos}kx_1\mathrm{Cos}\beta x_2$ gives a non-trivial solution. In the damped region we get a trivial answer for all possible combinations of elementary solutions. This suggests that the SH mode exists only as a harmonic wave. Taking $\mathrm{Cos}kx_1\mathrm{Cos}\beta x_2$ as the solution for the SH mode and normalizing it properly, in the sense of relation (2.8), one obtains the SH eigenmode as follows:

$$\vec{u}^{(5)}(\vec{x};k,\omega) = \frac{2}{\pi}\sqrt{\frac{(\beta^2 + k^2)}{\beta\,\omega}} \begin{bmatrix} 0 \\ 0 \\ \mathrm{Cos}kx_1\,\mathrm{Cos}\beta x_2 \end{bmatrix} \times \Theta(K_s - k). \tag{3.21}$$

3.2 Orthogonality

We have found five classes of eigenmodes for this problem. As the x_3 component of the displacement field occurs only in mode 5, this mode is orthogonal to all the other modes. Also mode 3 and 4 are each orthogonal to the other modes, due to the fact that the respective ranges of permitted values of k do not overlap. Finally, we find that although modes 1 and 2 have the same range of permitted values for k, they are also orthogonal to each other. The algebra to verify this is sketched below:

$$\langle \vec{u}^{(1)}(\vec{x},k,\omega),\ \vec{u}^{(2)}(\vec{x},k',\omega') \rangle = \int_0^\infty \int_0^\infty dx_1\, dx_2 \left[u_1^{(1)*}(\vec{x},k,\omega) \right.$$

$$\left. u_1^{(2)}(\vec{x},k',\omega') + u_2^{(1)*}(\vec{x},k,\omega)\, u_2^{(2)}(\vec{x},k',\omega') \right]$$

$$= \cdots \delta(k-k') \left[\frac{1}{(\alpha'^2 - \alpha^2)} (\alpha^2 + k^2) \right.$$

$$- \frac{1}{(\beta'^2 - \beta^2)} \frac{(\beta^2 - k^2)}{(\beta'^2 - k^2)} (\beta^2 + k^2)$$

$$\left. - \frac{2k^2}{(\beta'^2 - k^2)} \right]$$

$$= \cdots \delta(k-k') \left[\frac{(\alpha^2 + k^2)}{(\alpha'^2 - \alpha^2)} - \frac{(\beta^2 + k^2)}{(\beta'^2 - \beta^2)} \right]$$

$$= \cdots \delta(k-k') \left[\frac{\omega^2}{(\omega'^2 - \omega^2)} - \frac{\omega^2}{(\omega'^2 - \omega^2)} \right]$$

$$= 0.$$

Thus, we have an orthogonal set of modes.

3.3 The completeness relation

In this section we sketch the proof that the eigenmodes constructed in the previous section form a complete set. The completeness relation (2.9) takes the following form for $i = j = 1$.

$$\oint_m u_1^{(m)}(\vec{x};k,\omega)\, u_1^{(m)}(\vec{x}';k,\omega) = \delta(\vec{x}-\vec{x}') \ . \qquad (3.22)$$

Writing out the left hand side of the above equation explicity in terms of the eigenmodes described in the previous section one has four terms. In the first term changing the order of integration in ω and k and in the last term carrying out the k integration followed by the change of variable $\omega \longrightarrow C_r K_r$, we have after some algebraic rearrangements:

$$\oint_m u_1^{(m)}(\vec{x};k,\omega)\, u_1^{(m)}(\vec{x}',k,\omega) = \frac{2}{\pi^2} \int_0^\infty dk\ \mathrm{Sin}kx_1\, \mathrm{Sin}kx_1'$$

$$\left[\int_{kC_p}^{\infty} \frac{d\omega}{\omega} \frac{k^2}{\alpha} \cos\alpha(x_2 - x_2') \right.$$

$$+ \int_{kC_s}^{\infty} \frac{d\omega}{\omega} \beta \cos\beta(x_2 - x_2')$$

$$- \int_{kC_p}^{\infty} \frac{d\omega}{\omega} \frac{k^2}{\alpha} \frac{(\beta^2 - k^2)^2 - 4k^2\alpha\beta}{(\beta^2 - k^2)^2 + 4k^2\alpha\beta} \cos\alpha(x_2 + x_2')$$

$$+ \int_{kC_s}^{kC_p} \frac{d\omega}{\omega} \frac{k^2}{\Gamma} \frac{8k^2\Gamma(\beta^2 - k^2)^2\beta}{(\beta^2 - k^2)^4 + 16k^4\Gamma^2\beta^2} e^{-\Gamma(x_2 + x_2')}$$

$$+ \int_{kC_p}^{\infty} \frac{d\omega}{\omega} \beta \frac{(\beta^2 - k^2)^2 - 4k^2\alpha\beta}{(\beta^2 - k^2)^2 + 4k^2\alpha\beta} \cos\beta(x_2 + x_2')$$

$$+ \int_{kC_s}^{kC_p} \frac{d\omega}{\omega} \beta \frac{(\beta^2 - k^2)^4 - 16k^4\Gamma^2\beta^2}{(\beta^2 - k^2)^4 + 16k^4\Gamma^2\beta^2} \cos\beta(x_2 + x_2')$$

$$+ \int_{kC_s}^{kC_p} \frac{d\omega}{\omega} \beta \frac{8k^2\Gamma\beta(\beta^2 - k^2)^2}{(\beta^2 - k^2)^4 + 16k^4\Gamma^2\beta^2} \sin\beta(x_2 + x_2')$$

$$+ \int_{kC_p}^{\infty} \frac{d\omega}{\omega} \beta \frac{4k^2(\beta^2 - k^2)}{(\beta^2 - k^2)^2 + 4k^2\alpha\beta}$$

$$\{\cos(\alpha x_2 + \beta x_2') + \cos(\alpha x_2' + \beta x_2)\}$$

$$+ \int_{kC_s}^{kC_p} \frac{d\omega}{\omega} \beta \frac{16k^4\Gamma\beta(\beta^2 - k^2)}{(\beta^2 - k^2)^4 + 16k^4\Gamma^2\beta^2}$$

$$\{e^{-\Gamma x_2}\ \mathrm{Sin}\beta x_2' + e^{-\Gamma x_2'}\ \mathrm{Sin}\beta x_2\}$$

$$+ \int_{kC_s}^{kC_p} \frac{d\omega}{\omega}\ \beta\ \frac{4k^2(\beta^2-k^2)^3}{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}$$

$$\{e^{-\Gamma x_2}\ \mathrm{Cos}\beta x_2' + e^{-\Gamma x_2'}\ \mathrm{Cos}\beta x_2\}\Bigg]$$

$$+ \frac{1}{\pi}\int_0^\infty dK_r\ \mathrm{Sin}K_r x_1\ \mathrm{Sin}K_r x_1'\ \frac{(\Delta_r^2+K_r^2)^2}{R(K_r)}\ \Delta_r$$

$$\Bigg[\ K_r^2\ e^{-\Gamma_r(x_2+x_2')} + \frac{4K_r^2\Gamma_r^2\Delta_r^2}{(\Delta_r^2+K_r^2)_2}\ e^{-\Delta_r(x_2+x_2')}$$

$$- \frac{2K_r^2\Gamma_r\Delta_r}{(\Delta_r^2+K_r^2)}\ \left\{e^{-(\Delta_r x_2+\Delta_r x_2')}+e^{-(\Delta_r x_2'+\Delta_r x_2}\right\}\Bigg]. \qquad (3.23)$$

Here

$$\Gamma_r = \sqrt{K_r^2-K_p^2} \quad \text{and} \quad \Delta_r = \sqrt{K_r^2-K_s^2}\ .$$

In the above equation after a second change of variable from ω to $\alpha(=\sqrt{\frac{\omega^2}{C^2}-k^2})$ in the first term and from ω to $\beta(=\sqrt{\frac{\omega^2}{C_s^2}-k^2})$ in the second term, we have those two terms as follows:

$$\frac{2}{\pi^2}\int_0^\infty dk\ \mathrm{Sin}kx_1\ \mathrm{Sin}kx_1'\left[\int_0^\infty d\alpha\ \frac{k^2}{(\alpha^2+k^2)}\ \mathrm{Cos}\alpha(x_2-x_2')\right.$$

$$+ \int_0^{\infty} d\beta \, \frac{\beta^2}{(\beta^2+k^2)} \, \mathrm{Cos}\beta(x_2-x_2') \Bigg] . \qquad (3.24)$$

These terms lead to the desired delta function appearing in the right hand side of equation (3.22), so that in order to proof completeness, the remaining terms in equation (3.23) must cancel each other. Collecting the terms involving $\alpha(x_2+x_2')$ and its analytic continuations we have:

$$\frac{2}{\pi^2} \int_0^{\infty} dk \, \mathrm{Sin}kx_1 \, \mathrm{Sin}kx_1' \Bigg\{ - \int_{kC_p}^{\infty} \frac{d\omega}{\omega} \frac{k^2}{\alpha} \frac{(\beta^2-k^2)^2 - 4k^2\alpha\beta}{(\beta^2-k^2)^2 + 4k^2\alpha\beta} \, \mathrm{Cos}\alpha(x_2+x_2')$$

$$+ \int_{kC_s}^{kC_p} \frac{d\omega}{\omega} \frac{k^2}{\Gamma} \frac{8k^2\Gamma(\beta^2-k^2)^2\beta}{(\beta^2-k^2)^4 + 16k^4\Gamma^2\beta^2} \, e^{-\Gamma(x_2+x_2')} \Bigg]$$

$$+ \frac{1}{\pi} \int_0^{\infty} dK_r \, \mathrm{Sin}K_r x_1 \, \mathrm{Sin}K_r x_1' \, \frac{(\Delta_r^2+K_r^2)^2}{R(K_r)} \, \Delta_r$$

$$K_r^2 \, e^{-\Gamma_r(x_2+x_2')} . \qquad (3.25)$$

Rewriting the above expression we get:

$$\frac{1}{\pi^2} \int_0^{\infty} dk \, \mathrm{Sin}kx_1 \, \mathrm{Sin}kx_1'$$

$$\left[\int_{kC_p}^{\infty} \frac{k^2}{\alpha} \frac{4k^2\alpha\beta - (\beta^2 - k^2)^2}{(\beta^2 - k^2)^2 + 4k^2\alpha\beta} \left[e^{i\alpha(x_2+x_2')} + e^{-i\alpha(x_2+x_2')} \right] \right.$$

$$+ \int_{kC_s}^{kC_p} \frac{d\omega}{\omega} \frac{k^2}{i\Gamma} \left\{ \frac{4k^2 i\Gamma\beta - (\beta^2 - k^2)^2}{(\beta^2 - k^2)^2 + 4k^2 i\Gamma\beta} \right.$$

$$\left. \left. + \frac{4k^2 i\Gamma\beta + (\beta^2 - k^2)^2}{(\beta^2 - k^2)^2 - 4k^2 i\Gamma\beta} \right\} e^{-\Gamma(x_2+x_2')} \right]$$

$$+ \frac{1}{\pi} \int_0^{\infty} dK_r \, \mathrm{Sin}K_r x_1 \, \mathrm{Sin}K_r x_1' \, \frac{(\Delta_r^2 + K_r^2)^2}{R(K_r)} \Delta_r$$

$$K_r^2 \, e^{-\Gamma_r(x_2+x_2')} \, . \qquad (3.26)$$

Extending ω to the complex ξ ($=\omega+is$) plane we can convert the integral over ω to a contour integral and then use the residue theorem. First however, we have to introduce in the complex ξ plane two branch cuts, arising due to α ($= \sqrt{\frac{\omega^2}{C_p^2} - k^2}$) and β ($= \sqrt{\frac{\omega^2}{C_s^2} - k^2}$), as shown in Figure (3.1).

We define the square root functions

$$\zeta = \sqrt{\frac{\xi^2}{C_p^2} - k^2} \qquad \text{and} \qquad \eta = \sqrt{\frac{\xi^2}{C_s^2} - k^2} \qquad (3.27)$$

such that on the upper edge of the branch cuts for $\xi \geq C_p k$, $\zeta = \alpha$ and $\eta = \beta$ while for $C_s k \leq \xi \leq C_p k$, $\zeta = i\Gamma$ and $\eta = \beta$. Furthermore, on the lower edge of the branch cuts for $\xi \geq C_p k$, $\zeta = -\alpha$ and $\eta = -\beta$ while for $C_s k \leq \xi \leq C_p k$, $\zeta = i\Gamma$ and $\eta = -\beta$. With this choice of functions, the first four terms

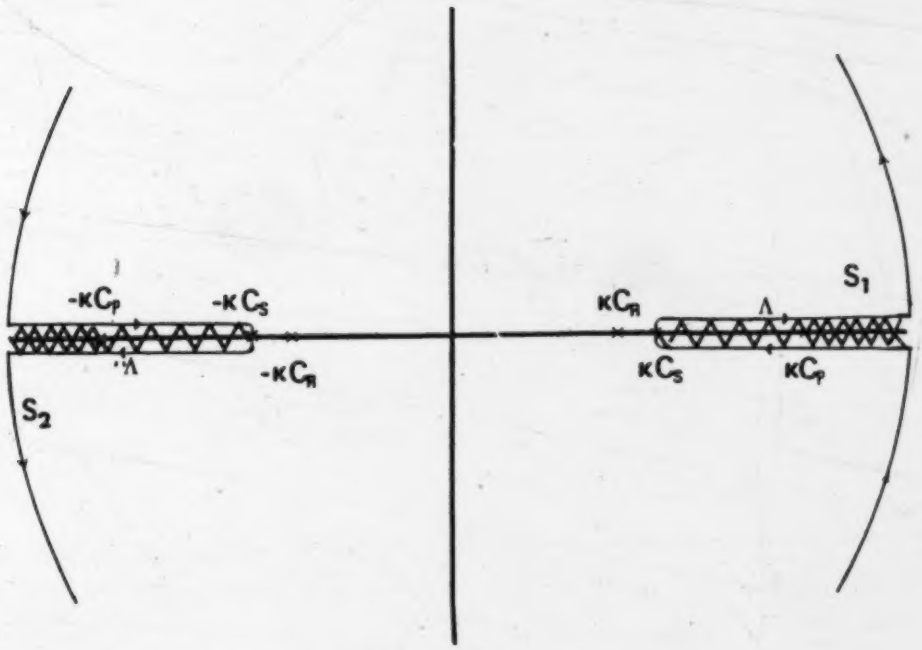

Figure 3.1 The branch cuts in the ξ plane and the path of integration Λ. The semicircles S_1 and S_2 will tend to ∞.

in equation (3.26) can be combined to yield an integral around the branch cuts, or two line integral running above and below the branch cuts along the real axis along the contour Λ:

$$\frac{1}{2\pi^2}\int_0^\infty dk\ \mathrm{Sin}kx_1\ \mathrm{Sin}kx_1' \int_\Lambda \frac{d\xi}{\xi}\ \frac{k^2}{\zeta}\ \frac{4k^2\zeta\eta - (\eta^2-k^2)^2}{(\eta^2-k^2)^2+4k^2\zeta\eta} \times e^{i\zeta(x_2+x_2')} \quad . \tag{3.28}$$

To evaluate this integral following standard procedures we close the contour in the upper and lower half-plane by the semicircles S_1 and S_2 and evaluate the integral by the residue theorem. The only poles that contribute are at $\xi = \pm C_r k$ (the Rayleigh poles). The pole at $\xi = 0$ is a double pole and does not contribute. The other two poles occur as complex conjugate pairs and cancel each other's contributions. Using l'Hospital's rule one finds for the integral (3.28) the value

$$2\pi i\ \frac{1}{2\pi^2}\left[\frac{i8K_r^4\Gamma_r\Delta_r^2}{4\ R(K_r)}\right] 2\ e^{-\Gamma_r(x_2+x_2')} \quad . \tag{3.29}$$

Using the Rayleigh wave secular equation (3.18) this can be rewritten as

$$-\frac{1}{\pi}\left[\frac{(\Delta_r^2+K_r^2)^2}{R(K_r)}\ \Delta_r K_r^2\right] e^{-\Gamma_r(x_2+x_2')} \quad . \tag{3.30}$$

This contribution just cancels the remaining term in

equation (3.26).

Similarly, the terms with $\beta(x_2+x_2')$, $(\alpha x_2+\beta x_2')$ and $(\alpha x_2'+\beta x_2)$ contribute only at the Rayleigh pole. These contributions then cancel with the Rayleigh terms of equation (3.23). Details of the calculations are shown in Appendix II.

A similar analysis holds for the other values of i and j.

Thus, one has a complete set of eigenfunctions for the quarterspace with one boundary semi-rigid and the other stress-free. These eigenfunctions can be used to derive an exact expression for the displacement field arising due to any line source. An example of such a problem is solved in the next chapter.

4. Exact Synthetic Seismograms

4.1 Introduction

The computational procedure for generating an exact synthetic seismogram due to a line source imbedded in a quarterspace is presented here.

Because the Green's function gives the response due to a force at a point in space and time, it follows that for a source with arbitary behaviour in time, the resultant response can be obtained by merely convolving this Green's function with the given time function. Furthermore, as shown by Singh, Ben-Menaham and Vered (1973), a source with an arbitary spatial description can also be expressed in terms of couples, double couples, centers of compression or explosion etc., which in turn can be derived from the spatial delta function. Therefore, one can develop the response due to an arbitary source from this Green's function.

Using the complete set of eigenfunctions developed in the previous chapter, we write an exact expression for the Green's function for this problem. This turns out to be an expression involving a triple integral which can be evaluated analytically following a technique similar to Cagniard (1939). Therefore, the response evaluated here is exact.

In the next section the expression for the displacement field due to a line source in a quarterspace, which is bounded by one semi-rigid and another free surface, is derived in terms of the appropriate Green's function. Following that, the explicit evaluation of this Green's function in an algebraic form is carried out using a technique similar to Cagniard (1939). In the last section the numerical computation and discussion are presented.

4.2 Construction of the Green's dyadic

Let $\vec{V}(\vec{x},t)$ be the displacement field in an homogeneous isotropic quarterspace, defined by $0 \leq x_1 \leq \infty$, $0 \leq x_2 \leq \infty$ and $-\infty \leq x_3 \leq \infty$, due to a line source of strength F_0 and direction $\vec{n}$, located at $(\vec{x}',t')$. The equations of motion are:

$$\ddot{\vec{V}} = C_p^2 \nabla\{\nabla\cdot\vec{V}\} - C_s^2 \nabla\times\nabla\times\vec{V} + F_0\vec{n}\,\delta(\vec{x}-\vec{x}')\delta(t-t'). \qquad (4.1)$$

The displacement field is subject to the boundary conditions:

$$\vec{V}\cdot\vec{I}_1 = 0 \text{ and } \sigma_{12} = \sigma_{13} = 0 \text{ along the surface } x_1 = 0. \qquad (4.2)$$

$$\vec{\sigma}:\vec{I}_2 = 0 \quad \text{along the surface } x_2 = 0, \qquad (4.3)$$

Here C_p and C_s are the P- and S-wave velocities respectively, $\vec{n}$ is a unit vector along the force direction,

while F_o is a measure of the strength of the force and has dimension (Length^3/Time). $\vec{I}_1$ and $\vec{I}_2$ are unit vectors along the x_1 and x_2 axes respectively and $\overleftrightarrow{\sigma}$ is the stress dyadic.

After replacing t by $t=\tau+t'$ equation (4.1) can be rewritten as:

$$-\frac{\partial^2\vec{V}}{\partial\tau^2} + C_p^2\,\nabla\{\nabla\cdot\vec{V}\} - C_s^2\,\nabla\times\nabla\times\vec{V} = -F_o\vec{n}\,\delta(\vec{x}-\vec{x}')\delta(\tau). \quad (4.4)$$

Expressing the time behaviour of the displacement field by a Fourier integral:

$$\vec{V}(\vec{x},\vec{x}',\tau) = \int_{-\infty}^{+\infty} dp\; e^{-ip\tau}\,\vec{V}^{(p)}(\vec{x},\vec{x}'), \quad (4.5)$$

equation (4.4) takes the form:

$$p^2\,\vec{V}^{(p)}(\vec{x},\vec{x}') + C_p^2\,\nabla\{\nabla\cdot\vec{V}^{(p)}(\vec{x},\vec{x}')\} - C_s^2\,\nabla\times\nabla\times\vec{V}^{(p)}(\vec{x},\vec{x}')$$

$$= -F_o\vec{n}\,\delta(\vec{x}-\vec{x}') \; . \quad (4.6)$$

Clearly the solution $\vec{V}^{(p)}(\vec{x},\vec{x}')$ depends linearly on the applied force and one may write:

$$\vec{V}^{(p)}(\vec{x},\vec{x}') = \overleftrightarrow{G}^{(p)}(\vec{x},\vec{x}') : \vec{n}F_o \; . \quad (4.7)$$

Here $\overleftrightarrow{G}^{(p)}(\vec{x},\vec{x}')$ is the Green's dyadic function for this problem satisfying the following equation:

$$p^2 \overleftrightarrow{G}^{(p)}(\vec{x},\vec{x}') + C_p^2 \nabla\{\nabla\cdot\overleftrightarrow{G}^{(p)}(\vec{x},\vec{x}')\} - C_s^2 \nabla\times\nabla\times\overleftrightarrow{G}^{(p)}(\vec{x},\vec{x}')$$

$$= -\overleftrightarrow{I}\delta(\vec{x}-\vec{x}') \ .$$

In addition, $\overleftrightarrow{G}^{(p)}(\vec{x},\vec{x}')$ satisfies the same boundary conditions as $\vec{v}^{(p)}(\vec{x},\vec{x}')$, as stipulated by equations (4.2) and (4.3). Here $\overleftrightarrow{I}$ is the identity dyadic.

We may also rewrite the equation for $\overleftrightarrow{G}^{(p)}(\vec{x},\vec{x}')$ in component form:

$$p^2 G_{ij}^{(p)}(\vec{x},\vec{x}') + C_p^2 G_{kj,ki}^{(p)}(\vec{x},\vec{x}') - C_s^2 \left[\delta_{\ell k}\delta_{ni} - \delta_{\ell i}\delta_{nk}\right]\times$$

$$G_{\ell j,nk}^{(p)}(\vec{x},\vec{x}') = -\delta_{ij}\delta(\vec{x}-\vec{x}') \ .$$

On simplification this becomes:

$$p^2 G_{ij}^{(p)}(\vec{x},\vec{x}') + C_p^2 G_{kj,ki}^{(p)}(\vec{x},\vec{x}') - C_s^2 \left[G_{kj,ik}^{(p)}(\vec{x},\vec{x}')\right.$$

$$\left. - G_{ij,kk}^{(p)}(\vec{x},\vec{x}')\right] = -\delta_{ij}\delta(\vec{x}-\vec{x}') \ . \qquad (4.9)$$

Since we can always expand any function in terms of a complete set, the Green's function can be expanded in terms of the complete set of eigenfunctions derived in chapter 3. Thus, we have:

$$G_{ij}^{(p)}(\vec{x},\vec{x}') = \oint_n dk\, d\omega\, u_i^{(n)}(\vec{x},k,\omega)\, A^{(n)}(\vec{x}',k,\omega,p). \qquad (4.10)$$

Due to the properties of the eigenfunctions, the boundary conditions are automatically satisfied. Furthermore, in the previous chapter the completeness of these functions was shown and thus we have the relation:

$$\delta_{ij}\delta(\vec{x}-\vec{x}') = \sum_n \int dk\, d\omega\; u_i^{(n)}(\vec{x},k,\omega)\; u_j^{(n)}(\vec{x}',k,\omega). \qquad (4.11)$$

Substituting in equation (4.9) from equations (4.10) and (4.11) we have:

$$\sum_n \int dk\, d\omega\; (p^2-\omega^2)\; u_i^{(n)}(\vec{x},k,\omega)\; A^{(n)}(\vec{x}',k,\omega,p)$$

$$= -\sum_n \int dk\, d\omega\; u_i^{(n)}(\vec{x},k,\omega)\; u_j^{(n)}(\vec{x}',k,\omega)\;. \qquad (4.12)$$

From this we conclude that:

$$A^{(n)}(\vec{x}',k,\omega,p) = \frac{u_j^{(n)}(\vec{x}',k,\omega)}{(\omega^2-p^2)}\;. \qquad (4.13)$$

Replacing this value of $A^{(n)}(\vec{x}',k,\omega,p)$ in equation (4.10) one has:

$$G_{ij}^{(p)}(\vec{x},\vec{x}') = \sum_n \int dk\, d\omega\; \frac{u_i^{(n)}(\vec{x},k,\omega)\; u_j^{(n)}(\vec{x}',k,\omega)}{(\omega^2-p^2)}\;. \qquad (4.14)$$

Thus, we have finally an expression for the Green's function in terms of the known eigenfunctions. Hence, using equations (4.5) and (4.7) one can evaluate the displacement

field.

4.3 Evaluation of the Green's function

In this section we present an outline of the evaluation, in explicit form, of the Green's function which was derived in the previous section. As an example, the G_{zz} component is considered here. For the other components a similar analysis holds.

We may write the G_{zz} component as:

$$G_{zz}(\vec{x},\vec{x}',\tau) = \int_{-\infty}^{+\infty} dp\ e^{-ip\tau}\ G_{zz}^{(p)}(\vec{x},\vec{x}') \ .$$

Using equation (4.14) we then have:

$$G_{zz}(\vec{x},\vec{x}',\tau) = \int_{-\infty}^{+\infty} dp\ e^{-ip\tau} \sum_{n} \int dk\ d\omega \ \frac{u_z^{(n)}(\vec{x},k,\omega)\ u_z^{(n)}(\vec{x}',k,\omega)}{(\omega^2 - p^2)} \ . \qquad (4.15)$$

Here, in particular we consider the retarded Green's function i.e. the solution that is nonvanishing only for $\tau > 0$. We will be able to perform the integration along the real axis, if we will push the singularities arising from $(\omega^2-p^2)^{-1}$ into the lower half of the p-plane by adding an infinitesimal positive imaginary part ($i\epsilon$) to p. For $\tau > 0$, the p integral may be evaluated by closing the contour with a semicircle in the lower half plane and these poles

contributions are picked up. For $\tau < 0$, the p integral may be closed in the upper half plane and since in that region the integrand is analytic, the integral vanishes.

We write out the right hand side of equation (4.15) explicitly using the eigenfunctions defined in chapter 3. Following that we change the order of integration in ω and k for the first three terms in the summation in equation (4.15). For the last term in the summation, we first carry out the k integration and then change the variable ω to $C_r K_r$. After some algebraic rearrangements we find:

$$G_{22}(\vec{x},\vec{x}',\tau) = \frac{2}{\pi^3}\int_{-\infty}^{+\infty} e^{-ip\tau}\left[\int_0^{\infty} dk\ \mathrm{Cos}kx_1\ \mathrm{Cos}kx_1' \right.$$

$$\left[\int_{kC_p}^{\infty} \frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\ \alpha\ \mathrm{Cos}\alpha(x_2-x_2')\right.$$

$$+\int_{kC_s}^{\infty} \frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\ \frac{k^2}{\beta}\ \mathrm{Cos}\beta(x_2-x_2')$$

$$+\int_{kC_p}^{\infty} \frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\ \alpha\ \frac{(\beta^2-k^2)^2-4k^2\alpha\beta}{(\beta^2-k^2)^2+4k^2\alpha\beta}\ \mathrm{Cos}\alpha(x_2+x_2')$$

$$\left.\left.+\int_{kC_s}^{kC_p} \frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\ \Gamma\ \frac{8k^2\Gamma(\beta^2-k^2)^2\beta}{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}\ e^{-\Gamma(x_2+x_2')}\right.\right.$$

$$- \int_{kC_p}^{\infty} \frac{d\omega}{\omega\{\omega^2 - (p+i\epsilon)^2\}} \frac{k^3}{\beta} \frac{(\beta^2 - k^2)^2 - 4k^2\alpha\beta}{(\beta^2 - k^2)^2 + 4k^2\alpha\beta} \cos\beta(x_2 + x_2')$$

$$+ \int_{kC_s}^{kC_p} \frac{d\omega}{\omega\{\omega^2 - (p+i\epsilon)^2\}} \frac{k^3}{\beta} \frac{(\beta^2 - k^2)^4 - 16k^4\Gamma^2\beta^2}{(\beta^2 - k^2)^4 + 16k^4\Gamma^2\beta^2}$$

$$\cos\beta(x_2 + x_2')$$

$$- \int_{kC_s}^{kC_p} \frac{d\omega}{\omega\{\omega^2 - (p+i\epsilon)^2\}} \frac{k^3}{\beta} \frac{8\, k^2\Gamma(\beta^2 - k^2)^2\beta}{(\beta^2 - k^2)^4 + 16k^4\Gamma^2\beta^2}$$

$$\sin\beta(x_2 + x_2')$$

$$+ \int_{kC_p}^{\infty} \frac{d\omega}{\omega\{\omega^2 - (p+i\epsilon)^2\}} \alpha \frac{4k^2(\beta^2 - k^2)}{(\beta^2 - k^2)^2 + 4k^2\alpha\beta}$$

$$\{\cos(\alpha x_2 + \beta x_2') + \cos(\alpha x_2' + \beta x_2)\}$$

$$+ \int_{kC_s}^{kC_p} \frac{d\omega}{\omega\{\omega^2 - (p+i\epsilon)^2\}} \Gamma \frac{16k^4\Gamma(\beta^2 - k^2)\beta}{(\beta^2 - k^2)^4 + 16k^4\Gamma^2\beta^2}$$

$$\{e^{-\Gamma x_2} \cos\beta x_2' + e^{-\Gamma x_2'} \cos\beta x_2\}$$

$$- \int_{kC_s}^{kC_p} \frac{d\omega}{\omega\{\omega^2 - (p+i\epsilon)^2\}} \Gamma \frac{4k^2(\beta^2 - k^2)^3}{(\beta^2 - k^2)^4 + 16k^4\Gamma^2\beta^2}$$

$$\left.\{e^{-\Gamma x_2} \sin\beta x_2' + e^{-\Gamma x_2'} \sin\beta x_2\}\right]$$

$$+ \frac{\pi}{2} \int_0^{\infty} \frac{dK_r}{\{K_r^2 C_r^2 - (p+i\epsilon)^2\}} \cos K_r x_1 \cos K_r x_1' \frac{(\Delta_r^2 + K_r^2)}{R(K_r)} \Delta_r$$

$$\left\{ \Gamma_r^2 \, e^{-\Gamma_r(x_2+x_2')} + \frac{4K_r^2\Delta_r^2}{(\Delta_r^2+K_r^2)} \, e^{-\Delta_r(x_2+x_2')} - \frac{2K_r^2\Gamma_r\Delta_r}{(\Delta_r^2+K_r^2)} \left[e^{-(\Delta_r x_2+\Delta_r x_2')} + e^{-(\Delta_r x_2'+\Delta_r x_2)} \right] \right\} . \quad (4.16)$$

Here, the terms depending on the spatial coordinate x_1 and x_1' contain the product $\cos kx_1 \cdot \cos kx_1'$, which can be regarded as the sum of two cosine terms with arguments (x_1-x_1') and (x_1+x_1'). Taking into account also the terms dependence on x_2 and x_2', we find that the exponential terms in the integrand have only the following four combinations in the arguments:

1. (x_1-x_1') and (x_2-x_2'),
2. (x_1+x_1') and (x_2-x_2'),
3. (x_1-x_1') and (x_2+x_2'),
4. (x_1+x_1') and (x_2+x_2').

The first category represents waves travelling directly from the source to the receiver, the second represents waves reaching the receiver after interacting with the side (or vertical surface $x_1 = 0$), the third represents waves reaching the receiver after interacting with the top (or horizontal surface $x_2 = 0$), and finally the fourth type represents waves reaching the receiver after interacting with both the horizontal and the vertical surfaces. In the last category no distinction is made for the order of interaction with the

boundary surfaces. This is because both waves that are incident first on the vertical surface and then on the horizontal surface, or the other way around, are represented by the same term in equation (4.16). In all of the above categories the first arrivals are the waves following the geometrical path. These are followed by non-geometrical waves.

On the other hand, the terms dependent on the wavenumbers $\alpha(=\sqrt{K_p^2-k^2})$ and $\beta(=\sqrt{K_s^2-k^2})$, or their analytic continuation, appear with x_2 and x_2' in the exponents. The terms associated with α can be regarded as waves travelling with the P-wave velocity while those with β correspond to waves travelling with the S-wave velocity. On this basis one can improve the above classification. For each of the above categories there are terms in which the same wavenumber-dependent expression (α or β) appears before both x_2' and x_2. This suggests that there are waves travelling with both P- and S-wave velocities in each of the above categories. This leads to eight different terms.

Moreover for categories 3 and 4 there are terms which have different wavenumber-dependent terms associated with x_2' and x_2 in the exponents. Both of the combinations $(\alpha x_2'+\beta x_2)$ and $(\beta x_2'+\alpha x_2)$ occur for each of these categories. So we have four additional terms. These contributions can be regarded as converted waves which arrive at the horizontal surface, either directly from the source or after interacting with the vertical surface. They

arrive at the horizontal surface with the P- (or S-) wave velocity and after interacting go to the receiver with S- (or P-) wave velocity. Since the horizontal surface is stress-free, such mode conversions actually occur there.

On this basis the total field expressed by equation (4.16) can be grouped into twelve contributions. In the following table a special notation for these contributions is introduced.

Class	Contributions
I	P, S
II	PvP, SvS
III	PhP, PhS, ShS, ShP
IV	PvPhP (or PhPvP)
	PvPhS (or PhSvS)
	SvShS (or ShSvS)
	SvShP (or ShPvP)

Here h and v stand for interaction of the waves with the horizontal and vertical surfaces respectively. For example, SvShP stands for a wave which starts at the source as an S-wave and reaches the receiver as a P-wave after being reflected first at the vertical surface as an S-wave and then converted into a P-wave after reflection at the horizontal surface. In class IV all contributions have two interactions in their path from the source to the receiver, one at each of the horizontal and vertical surfaces. The

interactions can take place in any order and both such modes are included in the same term of equation (4.16). Because two such modes are indistinguishable in equation (4.16), either notation represents both of them.

For each of the contributions to equation (4.16) the evaluation of the triple integrals is carried out seperately. The method is outlined in the next section. It involves carrying out first the ω integration, followed by a Cagniard procedure for evaluating the remaining two integrals. The result is an algebraic expression which can be efficiently computed numerically. Summing all twelve contributions the total field is obtained.

4.4 Evaluation of the triple integral

The explicit evaluation of the multiple integral terms that occur in the expression of the components of Green's function, is sketched below. As an example the following term is considered:

$$\frac{2}{\pi^2}\int_{-\infty}^{+\infty} dp\; e^{-ip\tau}\left[\int_0^{\infty} dk\; \mathrm{Cosk}(x_1-x_1')\right.$$

$$\left.\int_{kC_p}^{\infty} \frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\;\alpha\;\frac{(\beta^2-k^2)^2-4k^2\alpha\beta}{(\beta^2-k^2)^2+4k^2\alpha\beta}\;\mathrm{Cos}\alpha(x_2+x_2')\right.$$

$$+\int_{kC_s}^{kC_p} \frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\,\beta\,\frac{8k^2\Gamma^2(\beta^2-k^2)^2}{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}\,e^{-\Gamma(x_2+x_2')}\Bigg]$$

$$+\frac{\pi}{2}\int_0^{\infty}\frac{dK_r}{\{K_rC_r-(p+i\epsilon)^2\}}\,\mathrm{Cos}K_r(x_1-x_1')\,\frac{(\Delta_r^2+K_r^2)}{R(K_r)}\,\Delta_r{}^x$$

$$\Gamma_r^2\,e^{-\Gamma_r(x_2+x_2')}\quad .\qquad (4.17)$$

This is the PhP contribution to the G_{22} component of the Green's function due to waves travelling with the P-wave velocity and reaching the receiver after interacting with the horizontal surface ($x_2=0$). The first term accounts for the harmonic part while the second term is the damped part. The last term is the Rayleigh wave contribution.

In order to evaluate this expression we first carry out the ω integration by lumping the three individual terms together as one term. After that, following Cagniard (1939), the remaining two integrals are played off against each other to the point of their mutual annihilation, whereby the solution is obtained directly.

The ω integration is performed exactly as in the proof of the completeness relation in the section 3.3. Namely, one notices that the first and second term of equation (4.17) are analytic continuations of each other and can be combined into a single integral along a closed path (same as the path Λ of section 3.3) in the complex ω-plane. Now this integral can be evaluated by applying the residue theorem. Only the poles that are at $\omega=\pm(p+i\epsilon)$ and $\omega=\pm C_rk$ give any

contribution, as the double pole at $\omega = 0$ does not give any contribution and remaining two poles are complex conjugate and cancel each others contributions. Furthermore, the contributions due to $\omega = \pm C_r k$ are exactly equal but opposite in sign to the third term of equation (4.17), and therefore cancel that term. Thus, after the ω integration is performed, only the residue contributions due to $\omega = \pm(p+i\epsilon)$ are left. After evaluating the residues and rescaling $k \rightarrow \frac{k}{C_p}$ the following expression is obtained:

$$\frac{i}{2\pi C_p^2} \int_{-\infty}^{+\infty} dp\, e^{-ip\tau} \int_0^{\infty} dk\, \mathrm{Cos}kX\, \frac{\sqrt{(p+i\epsilon)^2 - k^2}}{p^2} \times$$

$$\left[\frac{(p^2/\gamma^2 - 2k^2)^2 - 4k^4\sqrt{(p+i\epsilon)^2 - k^2}\sqrt{(p+i\epsilon)^2/\gamma^2 - k^2}}{(p^2/\gamma^2 - 2k^2)^2 + 4k^4\sqrt{(p+i\epsilon)^2 - k^2}\sqrt{(p+i\epsilon)^2/\gamma^2 - k^2}} \right] \times$$

$$e^{i\sqrt{(p+i\epsilon)^2 - k^2}\, Y} . \qquad (4.18)$$

Here $X = \frac{x_1 - x_1'}{C_p}$, $Y = \frac{x_1 + x_1'}{C_p}$, and $\gamma = \frac{C_s}{C_p}$.

The branch cuts in the complex k-plane run between $-\frac{p}{\gamma}$ and $\frac{p}{\gamma}$ as well as between $-p$ and p. The quantity $i\epsilon$, that appears inside the square roots helps in determining the proper signs for the square roots as well as in choosing the path of the k integration necessary for a retarded solution. For $p>0$, the path of integration should be taken below the cut while for $p<0$ it should be taken above the cut. The

following definitions are chosen for the square roots:

$$\sqrt{(p+i\epsilon)^2 - k^2} = \begin{cases} \mathrm{sgn}(p)\sqrt{p^2 - k^2} & k < |p| \\ i\sqrt{k^2 - p^2} & k > |p| \end{cases}$$

and

$$\sqrt{\frac{(p+i\epsilon)^2}{\gamma^2} - k^2} = \begin{cases} \mathrm{sgn}(p)\sqrt{\frac{p^2}{\gamma^2} - k^2} & k < |\frac{p}{\gamma}| \\ i\sqrt{k^2 - \frac{p^2}{\gamma^2}} & k > |\frac{p}{\gamma}| \end{cases}$$

for $\epsilon = 0$.

Using the transformation $k = |p|\ell$ one goes from the (p,k)-domain to the (p,ℓ)-domain. This transformation simplifies the integrand in equation (4.18) so that it becomes suitable for the Cagniard procedure. In the transformed ℓ-domain, the branch cuts run between $-\frac{1}{\gamma}$ and $\frac{1}{\gamma}$ and -1 and 1. The paths of integration in the ℓ-domain are below and above the cuts repectively for $p>0$ and $p<0$ (Figure 4.1). The square roots along the paths of integration assume the following value for $\epsilon = 0$:

$$\sqrt{(p+i\epsilon)^2 - k^2} = |p|L_1,$$

$$\sqrt{\frac{(p+i\epsilon)^2}{\gamma^2} - k^2} = |p|L_2$$

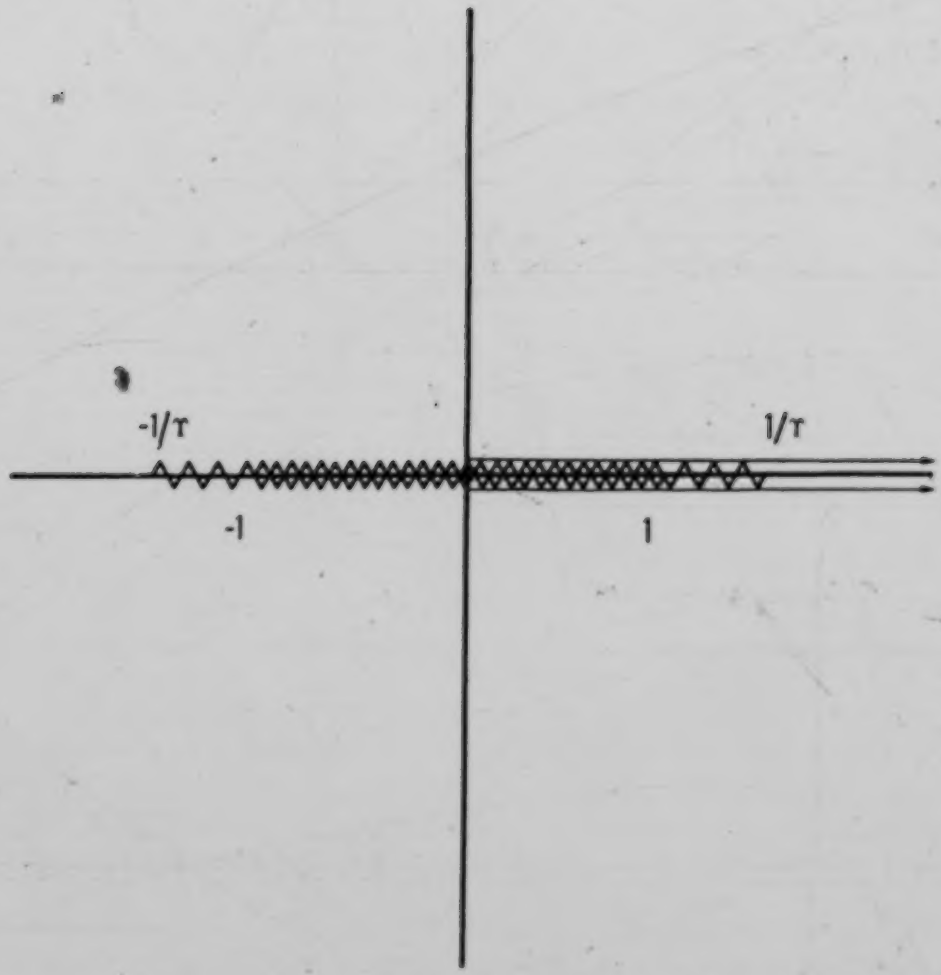

Figure 4.1 The branch cuts and paths of integration in the ℓ domain. The paths of integration are below and above the cuts respectively for $p>0$ and $p<0$.

where

$$L_1 = \begin{cases} \text{sgn}(p)\sqrt{1-\ell^2} & \ell < 1 \\ i\sqrt{\ell^2-1} & \ell > 1 \end{cases}$$

and

$$L_2 = \begin{cases} \text{sgn}(p)\sqrt{\frac{1}{\gamma^2}-\ell^2} & \ell < \frac{1}{\gamma} \\ i\sqrt{\ell^2-\frac{1}{\gamma^2}} & \ell > \frac{1}{\gamma} \end{cases}$$

Expression (4.17) takes now the form:

$$\frac{i}{2\pi C_p^2}\int_{-\infty}^{+\infty} dp\, e^{-ipr}\int_0^{\infty} d\ell\, \text{Cosp}\ell X\, e^{ipL_1 Y}\, G(\ell)\ , \qquad (4.19)$$

where

$$G(\ell) = L_1\,\frac{(1/\gamma^2-2\ell^2)^2-2\ell^2 L_1 L_2}{(1/\gamma^2-2\ell^2)^2+2\ell^2 L_1 L_2}\ . \qquad (4.20)$$

Expressing the cosine as a sum of two exponentials and splitting the p integration into the regions p>0 and p<0, four different terms are identified in equation (4.19). A typical term is:

$$\frac{i}{4\pi C_p^2} \int_0^\infty dp\, e^{-ip\tau} \int_0^\infty d\ell\, e^{ip(\ell X + L,Y)}\, G(\ell) \,. \qquad (4.21)$$

In order to evaluate this double integral following Cagniard (1939), a transformation from ℓ to $\tau^* = \ell x + L,Y$ is introduced. For this transformation to be unique, a branch cut is introduced in the complex τ^*-plane along the real axis between -R and R, where $R = \sqrt{X^2+Y^2}$. In the τ^*-plane the transformed path of integration (Λ^*) starts at the point Y below the cut and runs along the real axis up to R. After that it folds back and starts running above the cut up to the point X. From that point it takes off from the real axis and follows a curved path in the first quadrant which has the straight line through the origin and inclination $\mathrm{Tan}^{-1}(X/Y)$, as asymptote. The path Λ^* is closed in the first quadrant by adding a semi-circle S^* and a straight line path L^* (Figure 4.2). Eventually the radius of the semi-circle will be made to tend to ∞. Realizing that there are no singularities inside the closed path and that the integrand along the semi-circular path S^* tends to zero when S^* tends to ∞, we conclude that the expression (4.21) is equal to:

$$\frac{i}{4\pi C_p^2} \int_0^\infty dp\, e^{-ip\tau} \left[\int_Y^R + \int_R^\infty \right] d\tau^*\, e^{ip\tau^*} \left[\frac{d\ell}{d\tau^*}\, G(\ell) \right]_{\ell=\ell(\tau^*)} . \qquad (4.22)$$

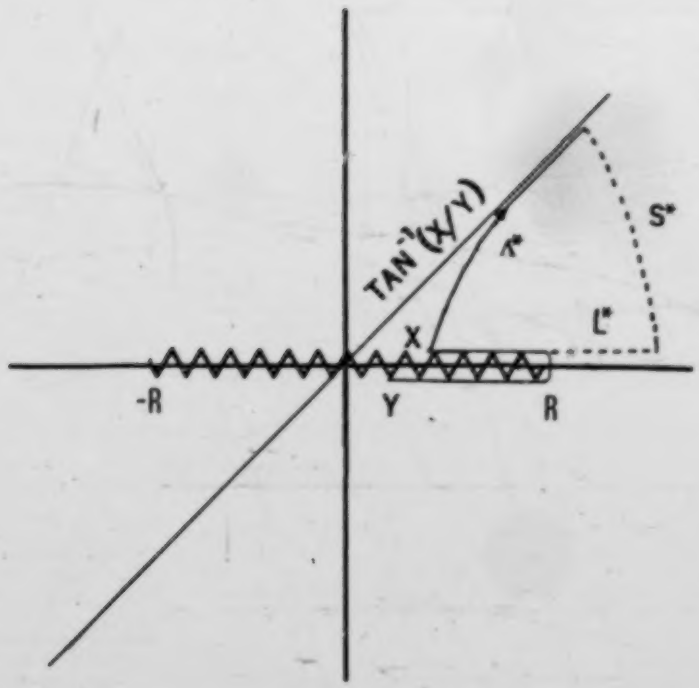

Figure 4.2 The branch cut and path of integration in the τ^* domain. The paths S^* and L^* are added at ∞.

In the τ^* plane the path of integration between Y and R is below the branch cut. After changing the order of integration, we perform the p integration obtaining a delta function term and a principal value part (in the sense of the theory of distributions):

$$\frac{i}{4\pi C_p^2}\left[\int_Y^R + \int_R^\infty\right] d\tau^* \left[\frac{d\ell}{d\tau^*} G(\ell)\right]_{\ell=\ell(\tau^*)} \left[\pi\delta(\tau - \tau^*) - iP\left[\frac{1}{\tau - \tau^*}\right]\right]. \tag{4.23}$$

Here the following identity was used.

$$\int_0^\infty dx\, e^{-ix(y-i\epsilon)} = \left[\frac{-i}{(y-i\epsilon)}\right] = \left[\pi\delta(y) - iP\left[\frac{1}{y}\right]\right]. \tag{4.24}$$

Similarly, carrying out the evaluation of the other three terms of expression (4.19), adding them together and performing some algebraic manipulation, we find that the left hand side of equation (4.17) yields

$$\frac{1}{2C_p^2} Im\left[\frac{d\ell}{d\tau} G(\ell)\right] + \tilde{P}\,. \tag{4.25}$$

Here Im stands for the imaginary part and

$$\ell(\tau) = \begin{cases} \dfrac{X\tau - iY\sqrt{\tau^2 - R^2}}{R^2} & \tau > R \\ \\ \dfrac{X\tau - Y\sqrt{R^2 - \tau^2}}{R^2} & Y \le \tau \le R. \end{cases} \tag{4.26}$$

$\tilde{P}$ stands for the principal value part of the corresponding terms.

This result is different from what is usually found in seismology. This is because the Fourier transform is used here for the time coordinate rather than the more usual Laplace transform which leads to a different splitting of the field involving only a delta function part and no principal value. In the usual analysis with the Cagniard procedure, using the Laplace transform, the principle of reflection is used to reduce the infinite range integral of inversion to a semi-infinite one. Mostlikely, due to this simplification the principal value part is lost.

All the principal value parts of equation (4.16) cancel each other when added together. Thus, if one ignores these principal value contributions, one obtains individual terms for the displacement field that diverge for large time. The sum of these terms which yields the net displacement field, however, is independent of neglecting the principal value terms and dies out for large times.

Since it is much simpler to ignore the principal value contributions and compute only the delta function contributions to the displacement field, we have ignored the principal value contributions in the numerical computations.

4.5 Numerical Results

For numerical computations we have placed a line source at a point (x_1', x_2') inside the quarterspace and calculated the response at regularly spaced points on the horizontal surface (x_2 =0). For a horizontal line source the Green's function components G_{11} and G_{21} determine the horizontal and the vertical components of the displacement field respectively. For a vertical line source the horizontal and the vertical components of the displacement field are given by the Green's function components G_{12} and G_{22} respectively.

The total field for each component of the Green's function can be decomposed into two groups; waves interacting with only the horizontal surface and waves interacting with both surfaces. The contributions of P, S, PhP, PhS, ShS and ShP, that belong to classes I and II, correspond to waves with an horizontal surface interaction, while the PvP, SvS, PvPhP, PvPhS, SvShS and SvShP contributions, of classes III and IV, give rise to waves interacting with both the horizontal and vertical surfaces.

The receivers are being placed at the horizontal surface (x_2=0), the distinction between terms involving ($x_2 \pm x_2'$) disappears. It follows therefore that the ray paths and hence the arrival times of the harmonic part of both unconverted and converted waves are the same as that of the direct wave. Thus, for each of the two groups mentioned above, there are only two distinct first arrivals,

corresponding to the P- and S-wave velocities respectively and consisting of the harmonic parts of the direct, unconverted and converted waves, all lumped together.

The surface wave contributions appear explicitly in equation (4.16), but in the course of ω integation for evaluating the integrals, these contributions were lumped together with all eight terms involving the unconverted and converted contributions that represent the interaction of the waves with the horizontal surface. Therefore we expect the contribution due to these terms to show, after the first arrivals, the appearance of the surface waves. This is indeed the case and such features are found on the seismograms.

The head waves are imbedded in the ShS and the ShP contributions of waves interacting only with the horizontal surface and in the SvShS and the SvShP contributions of waves interacting with both the vertical and horizontal surfaces. In each of these two groups there are six contributing terms containing P, S, head wave and Rayleigh wave pulses. However, depending upon the component of the Green's function at any particular location, the head and/or Rayleigh pulse may not be observed.

In Figures (4.3-4.6), for a source placed at the location (40km,20.0km) and the receiver at (100km,0.0km), all four components of the Green's function are shown. In each of them the first six graphs are the individual contributions to the first group (field interacting only

GREEN'S FUNCTION (1,1)
A display of few contributions
source Location : 40. km, 20. km
Receiver Location : 100. km, 0. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Time - sec
20.0
40.0
60.0
80.0
100.
Various Contributions
SHS + SHP
PHP + PHS
SHP
SHS
S
PHS
PHR
R

Figure 4.3

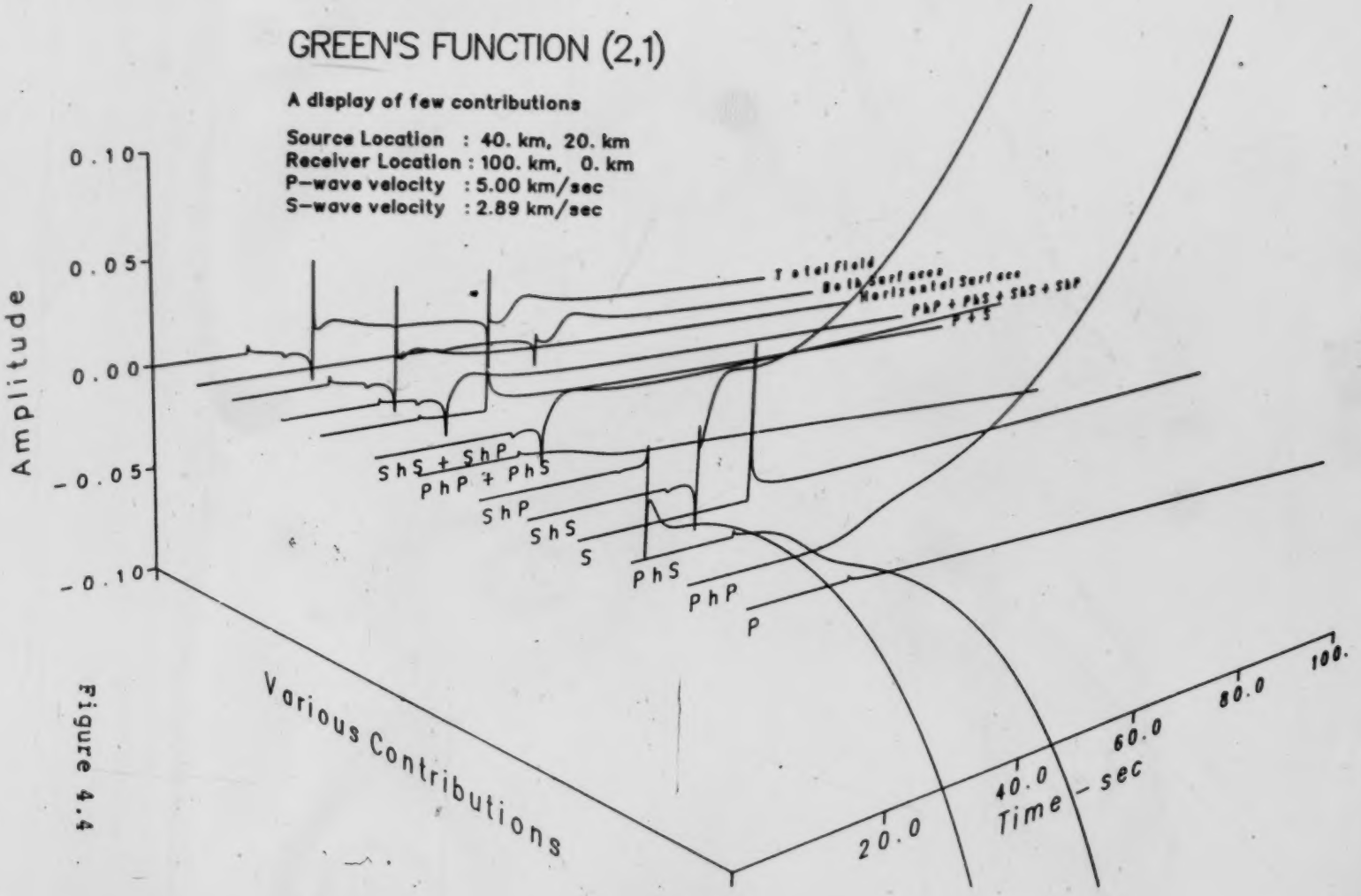
GREEN'S FUNCTION (2,1)
A display of few contributions
Source Location : 40. km, 20. km
Receiver Location : 100. km, 0. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Total Field
Both Surfaces
Horizontal Surface
PhP + PhS + ShS + ShP
P + S
ShS + ShP
PhP + PhS
ShP
ShS
S
PhS
PhP
P
Various Contributions
20.0
40.0
60.0
80.0
100.
Time - sec

Figure 4.4

GREEN'S FUNCTION (1,2)
A display of few contributions
Source Location : 40. km, 20. km
Receiver Location : 100. km, 0. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Total Field
Body + Total Surface
Horiz + Total Surface
PHP + PHS + SHS + SHP
P + S
SHS + SHP
PHP + PHS
SHP
SHS
S
PHS
PHP
P
Amplitude
0.10
0.05
0.00
-0.05
-0.10
0.0
40.0
60.0
80.0
100.
Time - sec
Various Contributions

Figure 4.5

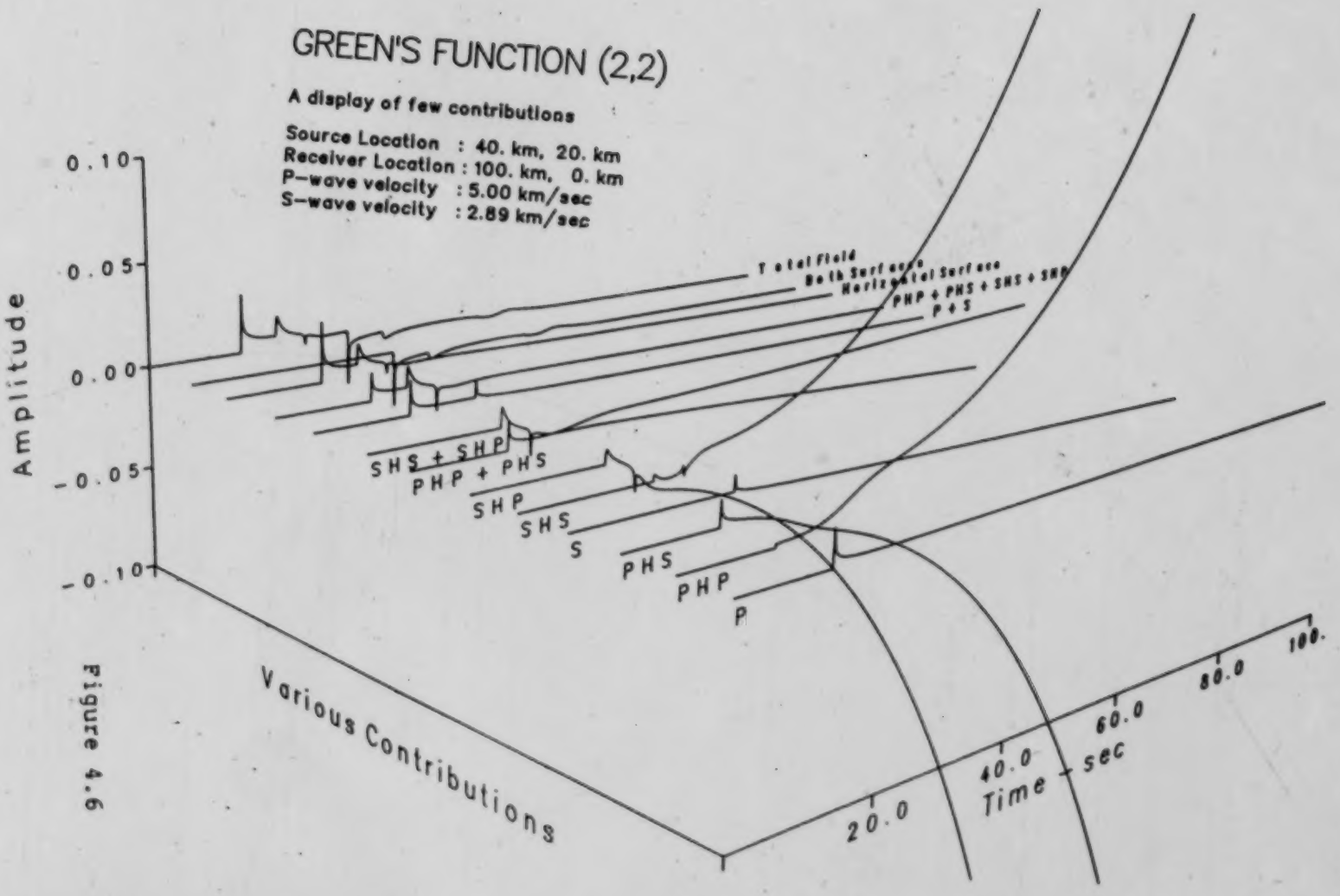
GREEN'S FUNCTION (2,2)
A display of few contributions
Source Location : 40. km, 20. km
Receiver Location : 100. km, 0. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Total Field
Both Surfaces
Horizontal Surface
PHP + PHS + SHS + SHP
P + S
SHS + SHP
PHP + PHS
SHP
SHS
S
PHS
PHP
P
Various Contributions
20.0
40.0
60.0
80.0
100.
Time - sec

Figure 4.6

with the horizontal surface).

The numerical values of the individual contributions seem to diverge as $t \rightarrow \infty$, although, the sum of all six contributions, which is the eleventh graph in these Figures, produces a well behaved signal which damps out for large values ot t. As explained before, the divergent behaviour for large times in these individual contributions are due only to the way we made the numerical calculation. Each of the contributions, as it is defined, in equation (4.16) (in the form of a triple integral) has a convergent behaviour as $t \rightarrow \infty$. The evaluation of these triple integrals yields an algebraic expressions from the delta function plus additional terms. Both types of terms contain a divergent part such that their sum is well behaved. Furthermore these additional divergent terms when added together for all six terms also cancel each other. Thus, as far as the final result is concerned, it is sufficient to calculate only the algebraic expression for each of the terms. Because it is simpler to calculate the algebraic expressions only, we only calculate these part and find the total displacement by adding them together. Thus, the individual contributions, as shown here, do not represent the true behaviour of these terms. The growth of these terms for large times of terms here is simply a numerical artifact. Moreover, in the literature (Aki and Richards 1980) one works with a different approach in which the individual contributions diverge for large times.

The P and S contributions come as sharp pulses at the P and S wave velocities respectively. As $t \longrightarrow \infty$ they diverge in opposite directions. These divergent parts correspond to pole contributions which were not taken into account while evaluating these terms numerically.

In the course of the ω integration for the P and S contributions one also obtains a contribution from a pole at $\omega = 0$. After the p integration this contribution gives rise to a term which is a linear function of time. Hence, the P and S contributions, as defined in equation (4.16), diverge as $t \longrightarrow \infty$. These extra terms for the P and S contributions are equal but opposite in sign. Graph nine in Figures (4.3-4.6), which shows the sum of the P and S contributions, displays this artificial behaviour. Thus, even though we have not taken into account these pole contributions the total field remaines uneffected.

The first arrivals of the harmonic part of the PhP and the PhS contributions are received at the P-wave velocity. These are followed by an inhomogeneous wave part which shows strong growth for large times. However, the resultant of the addition of the PhP and the PhS contribution shows slower growth. The sum of this resultant with the sum of the ShS and ShP dies out as $t \longrightarrow \infty$. In the ShS and ShP contributions the first arrivals are the head wave contributions followed by a sharp pulse at the S-wave velocity, followed by the inhomogeneous part. The sum of the inhomogeneous parts associated with the PhP, PhS, ShS and

ShP forms a well defined Rayleigh pulse.

The x_1, x_1' dependence in the Green's function components occurs through the combinations (x_1-x_1') and (x_1+x_1'). These represent waves without vertical surface interactions and waves with such interactions respectively. The transformation $x_1 \longrightarrow (x_1+2x_1')$ converts (x_1-x_1') into (x_1+x_1'). Thus, on the horizontal surface ($x_2=0$) the two groups of waves are actually of the same type. That is why in the above discussion of the individual contributions reference was made only to those contributions responsible for waves with horizontal surface interactions. The contributions of waves with both, horizontal and vertical surface interactions at a location (x_1), are the same as the waves observed with only horizontal surface interactions at the location (x_1+2x_1'). In Figures (4.7-4.10) we present the waves with only horizontal surface interactions and in Figures (4.11-4.14) we present the waves with both surface interactions for each of the four components of the Green's function. The translational relationship between the two groups of waves is obvious there. Also, a flip in polarity is seen, in the two groups, for the G_{11} and G_{21} components of the Green's functions. This polarity reversal is due to the fact that $Sinkx_1Sinkx_1'$ and $Coskx_1Sinkx_1'$ are contained respectively in the terms for G_{11} and G_{21}, which can be seen by expressing these terms as a sum of two trigonometric functions.

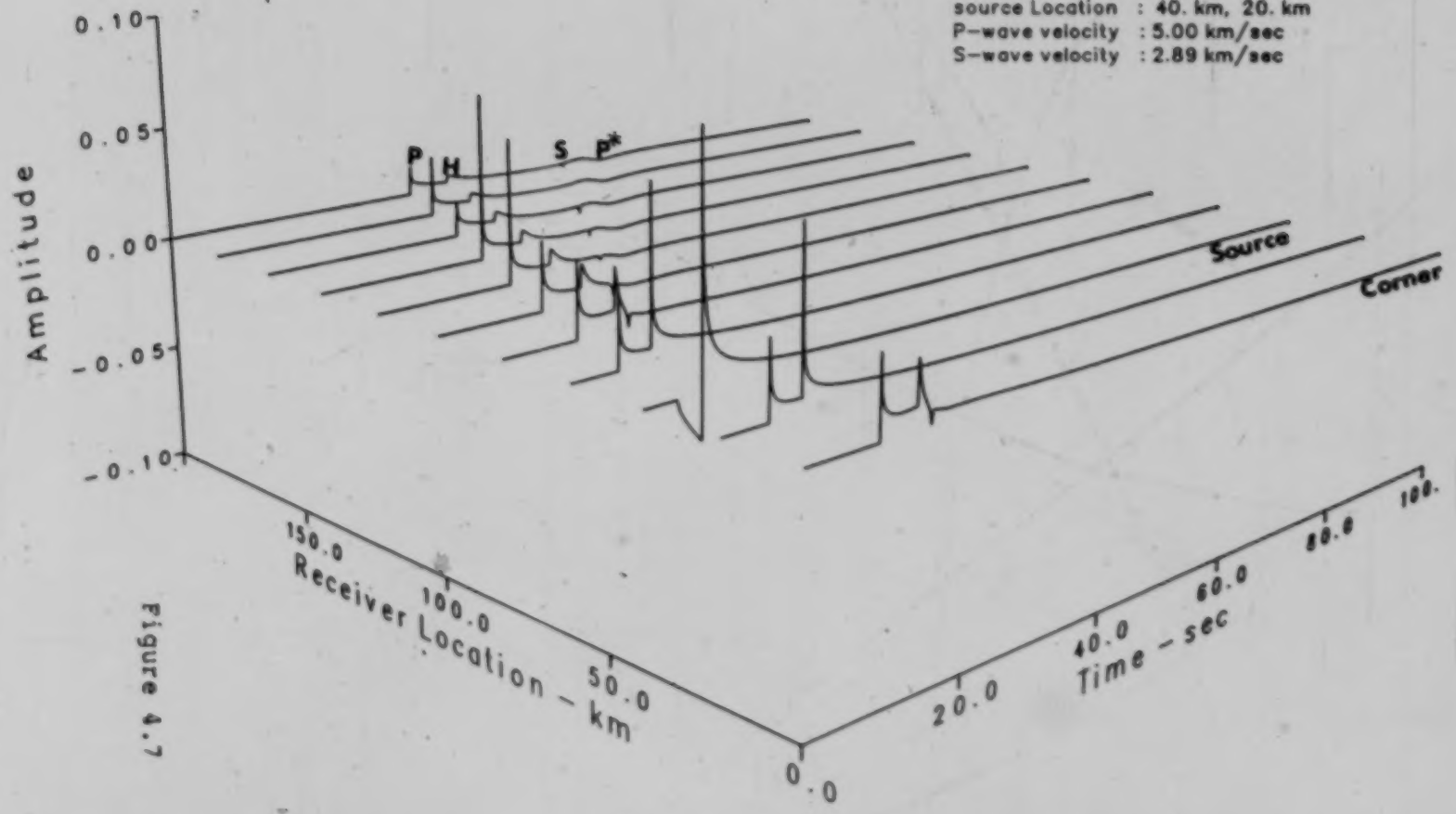
GREEN'S FUNCTION (1,1)
Field with horizontal surface interaction
source Location : 40. km, 20. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Amplitude
0.10
0.05
0.00
-0.05
-0.10
P
H
S
P*
Source
Corner
150.0
100.0
50.0
0.0
Receiver Location - km
20.0
40.0
60.0
80.0
100.
Time - sec

Figure 4.7

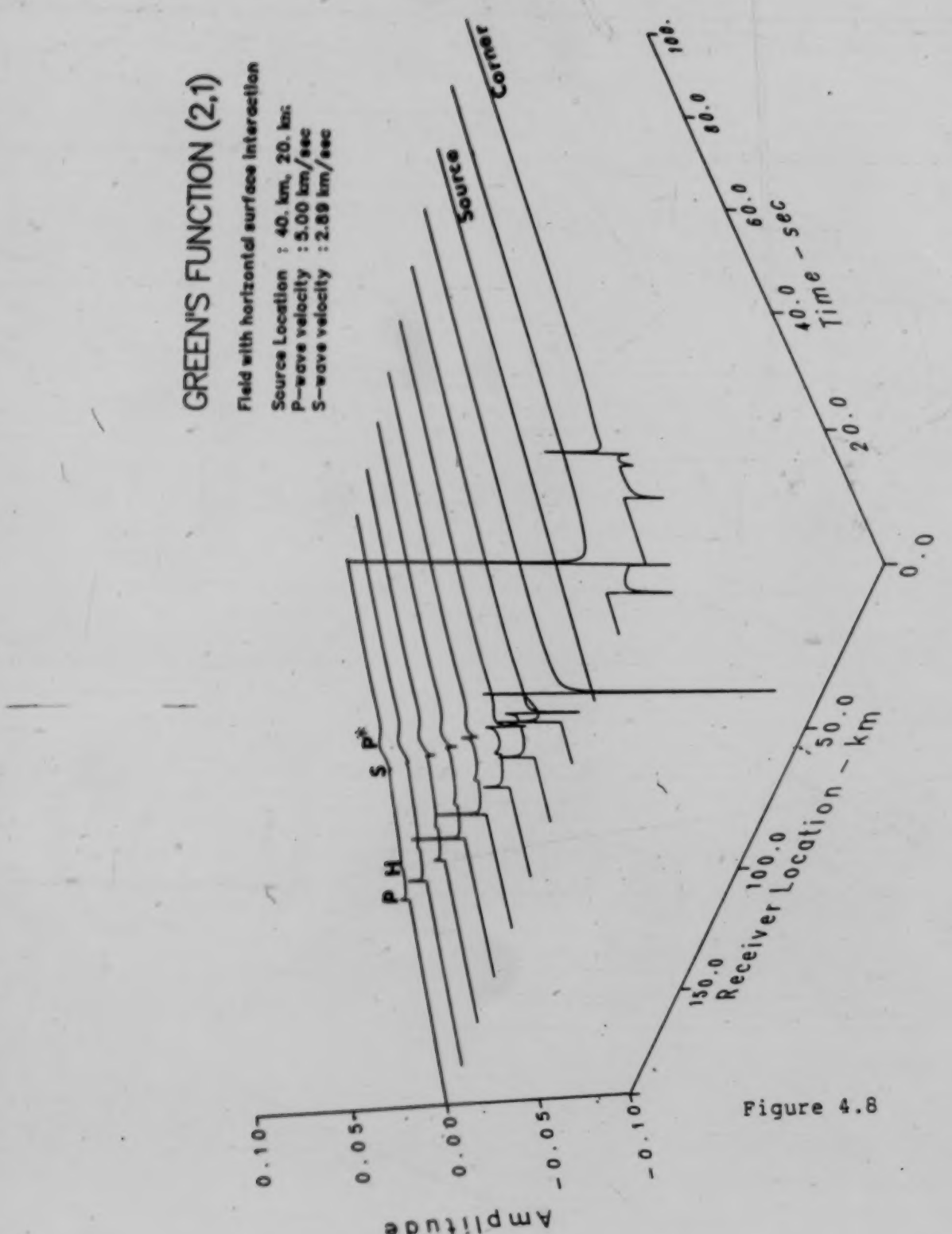
GREEN'S FUNCTION (2,1)
Field with horizontal surface interaction
Source Location : 40. km, 20. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Source
Corner
P H
S P
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Receiver Location - km
0.0
50.0
100.0
150.0
Time - sec
0.0
20.0
40.0
60.0
80.0
100.

Figure 4.8

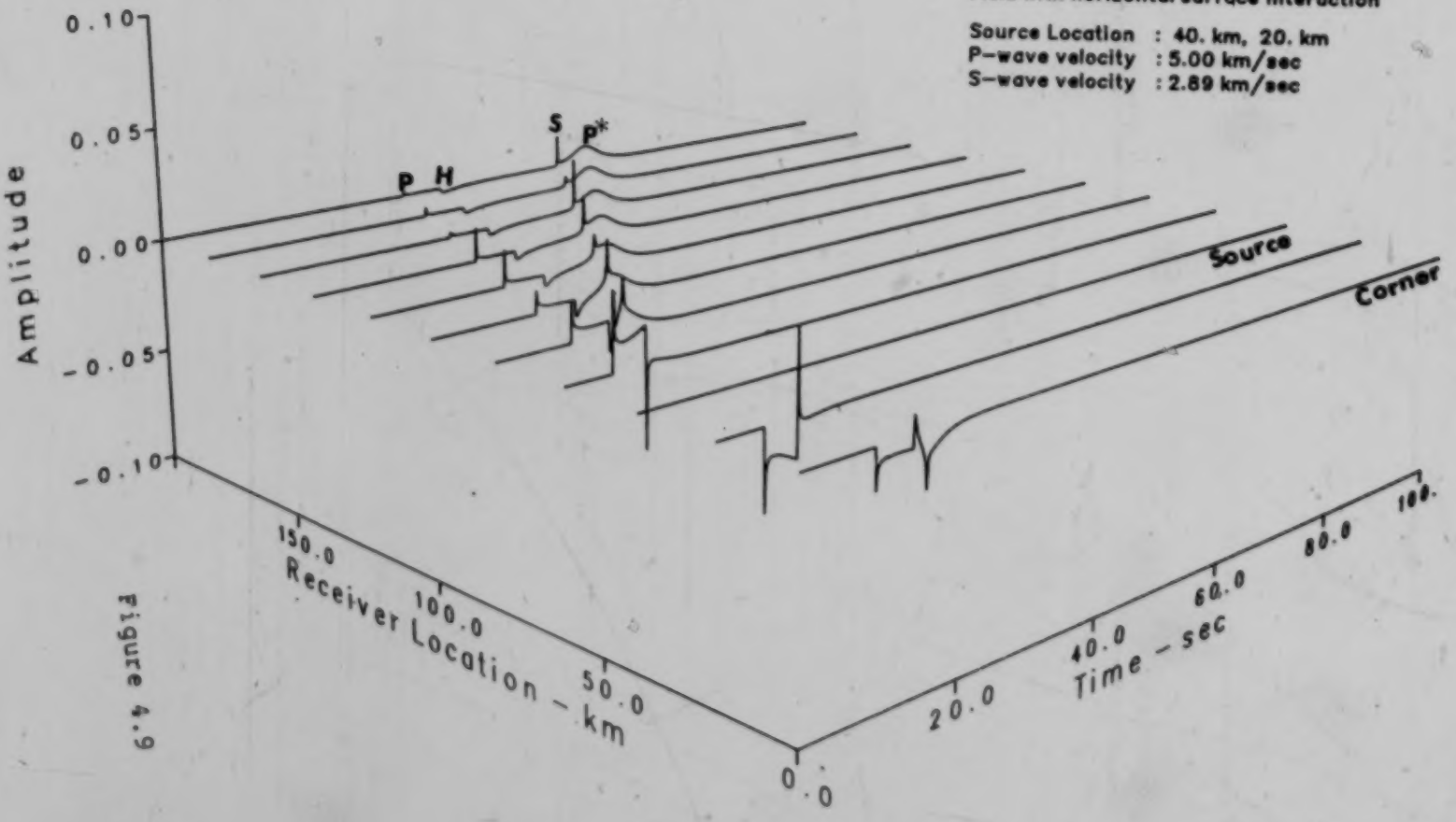
GREEN'S FUNCTION (1,2)
Field with horizontal surface interaction
Source Location : 40. km, 20. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Amplitude
0.10
0.05
0.00
-0.05
-0.10
P
H
S
P*
Source
Corner
150.0
100.0
50.0
0.0
Receiver Location - km
20.0
40.0
60.0
80.0
100.
Time - sec

Figure 4.9

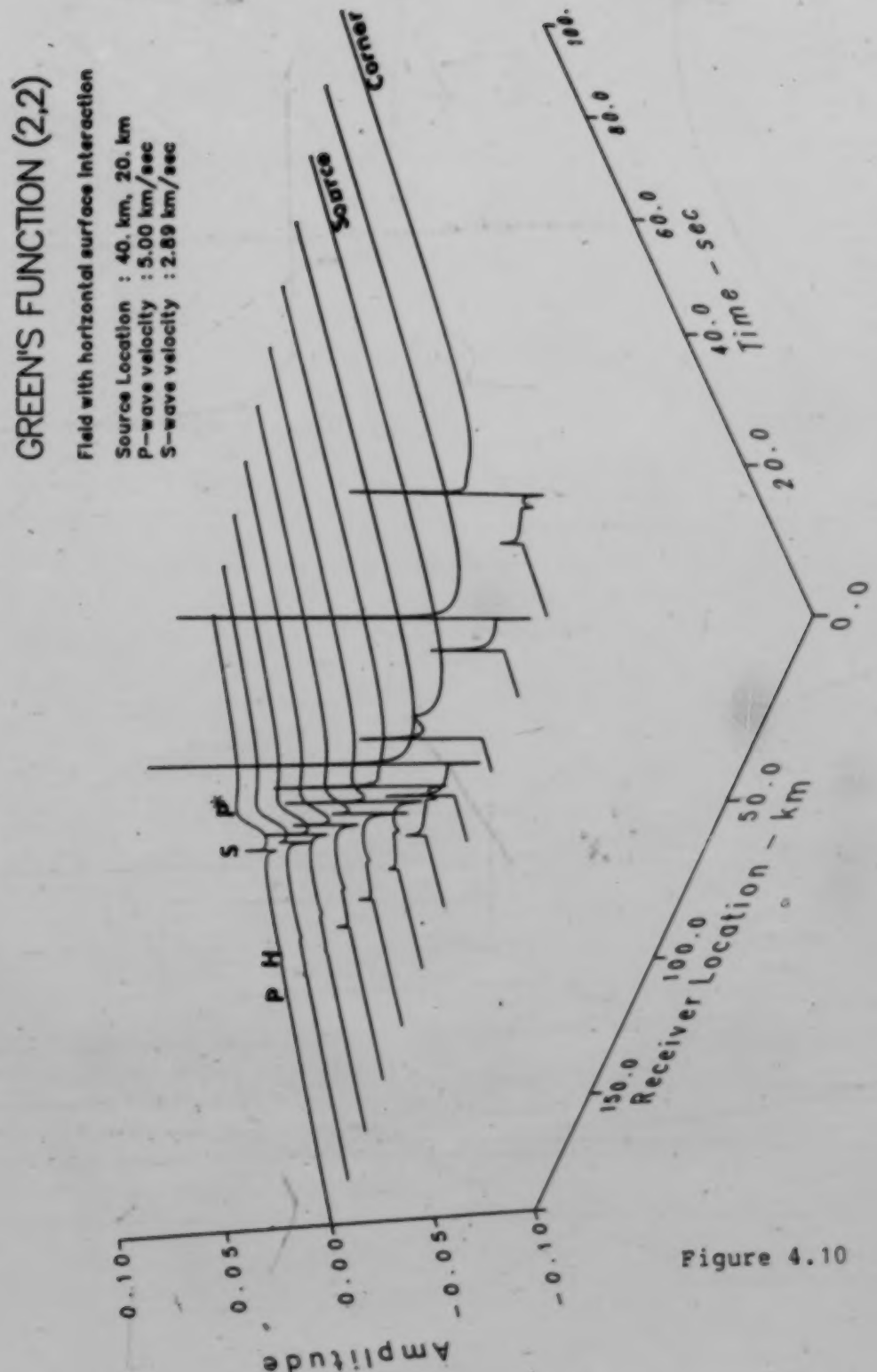
GREEN'S FUNCTION (2,2)
Field with horizontal surface interaction
Source Location : 40. km, 20. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Corner
Source
S P*
P H
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Receiver Location - km
0.0
50.0
100.0
150.0
Time - sec
20.0
40.0
60.0
80.0
100.

Figure 4.10

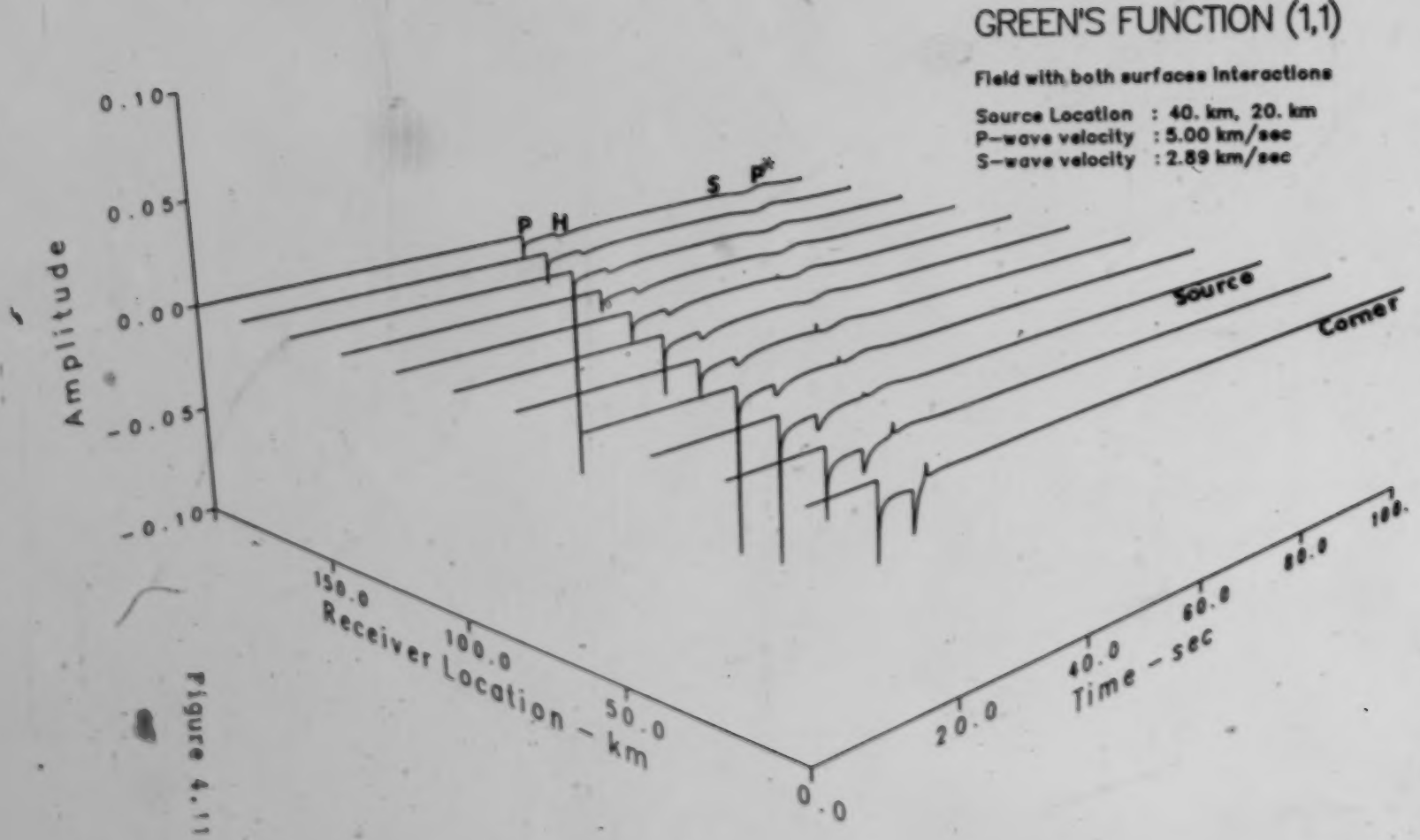
GREEN'S FUNCTION (1,1)
Field with both surfaces interactions
Source Location : 40. km, 20. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
P H
S P*
Source
Corner
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Receiver Location - km
150.0
100.0
50.0
0.0
Time - sec
20.0
40.0
60.0
80.0
100.

Figure 4.11

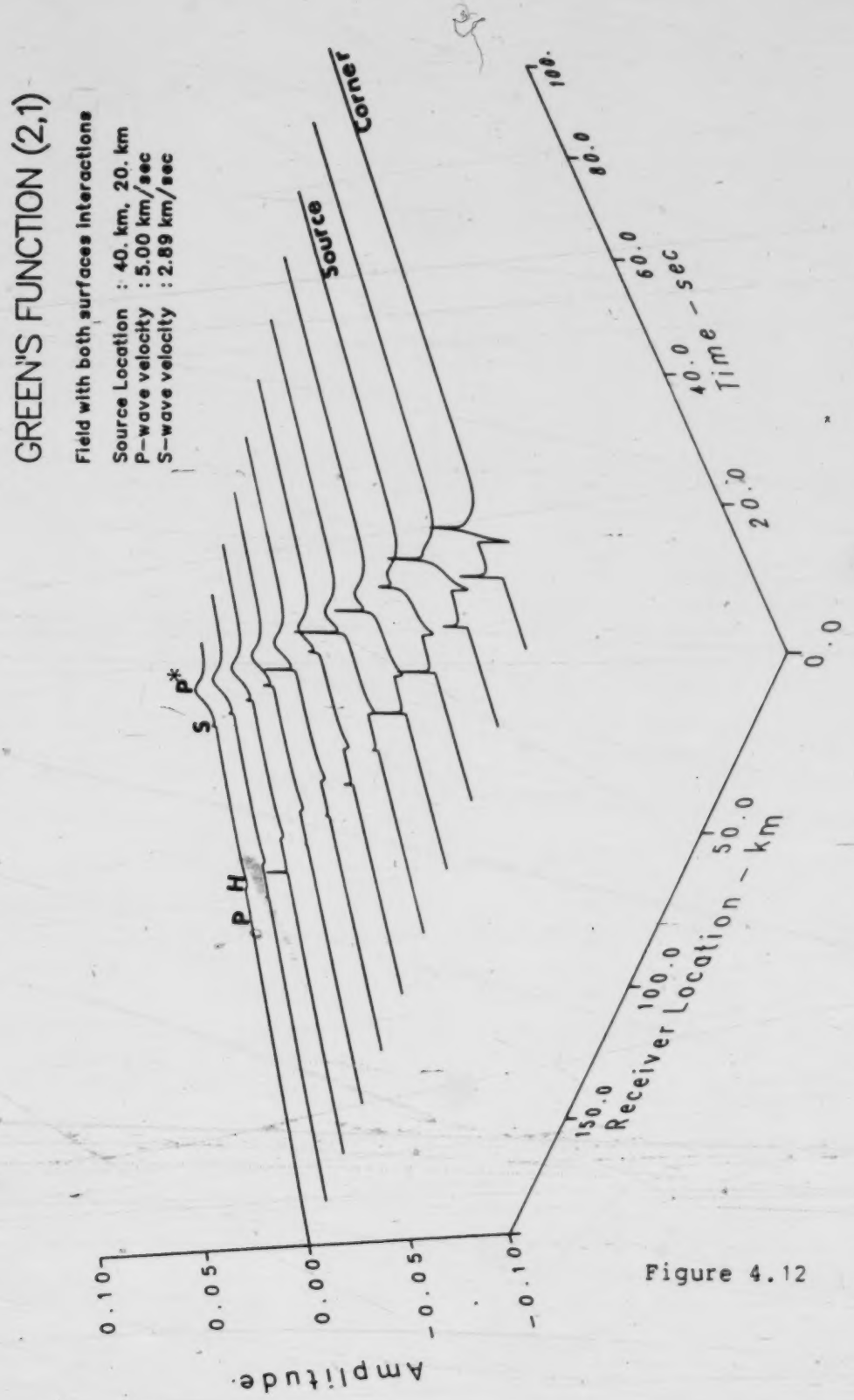
GREEN'S FUNCTION (2,1)
Field with both surfaces interactions
Source Location : 40. km, 20. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Corner
Source
S
P*
P
H
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Receiver Location - km
0.0
50.0
100.0
150.0
Time - sec
0.0
20.0
40.0
60.0
80.0
100.

Figure 4.12

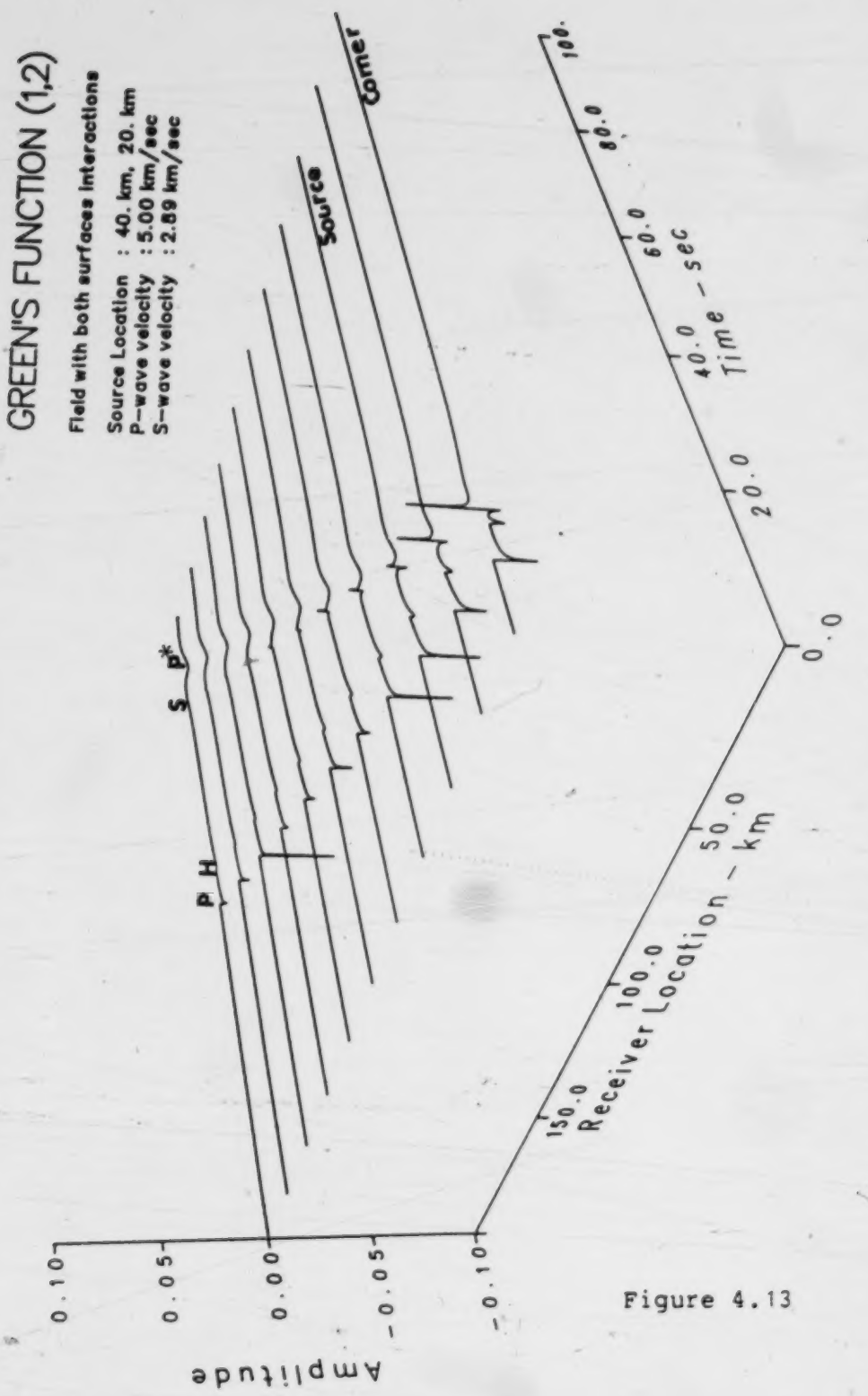
GREEN'S FUNCTION (1,2)
Field with both surfaces interactions
Source Location : 40. km, 20. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Corner
Source
S p*
P H
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Receiver Location - km
50.0
100.0
150.0
Time - sec
0.0
20.0
40.0
60.0
80.0
100.

Figure 4.13

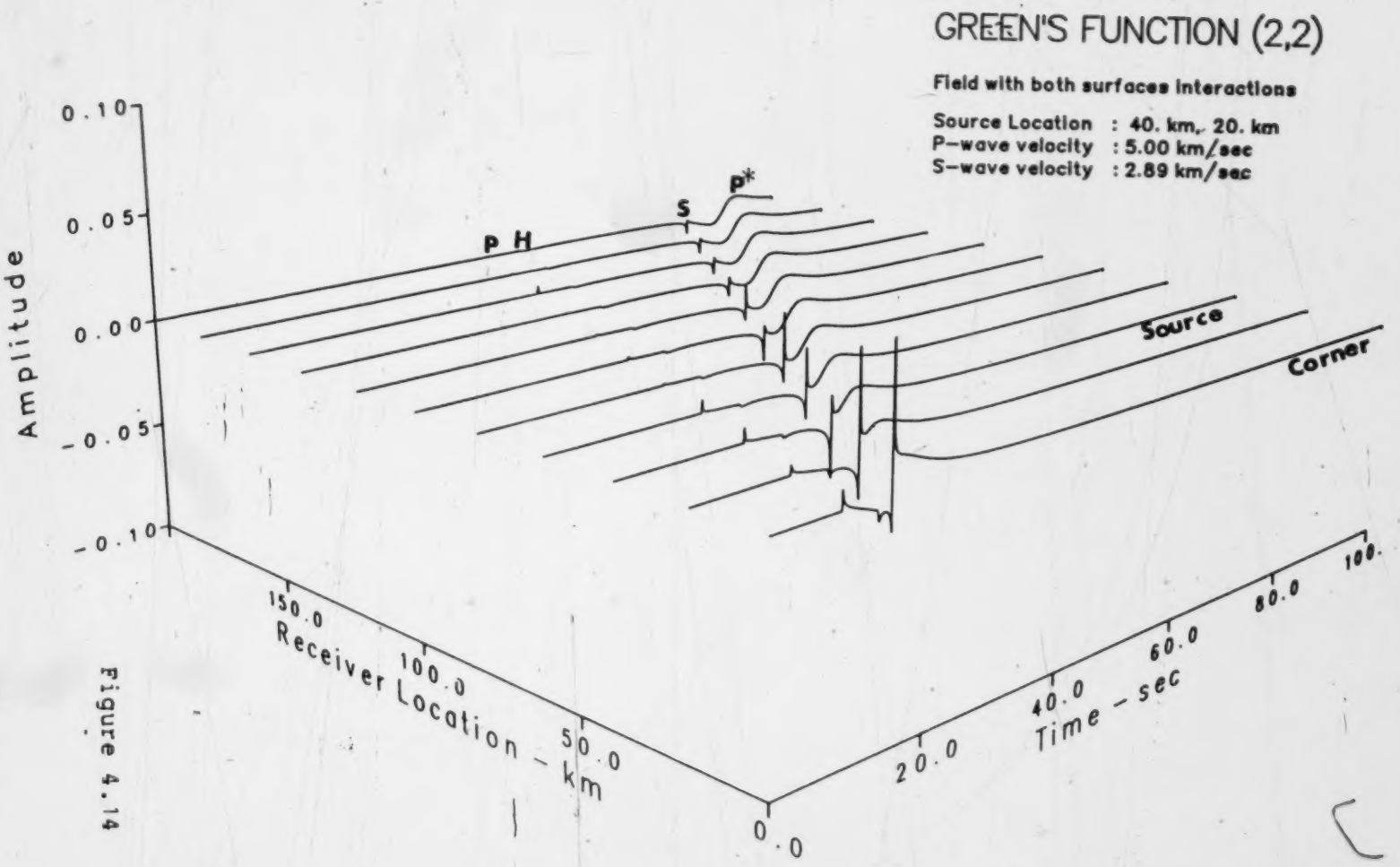
GREEN'S FUNCTION (2,2)
Field with both surfaces interactions
Source Location : 40. km, 20. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
P H
S
P*
Source
Corner
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Receiver Location - km
150.0
100.0
50.0
0.0
Time - sec
20.0
40.0
60.0
80.0
100.

Figure 4.14

Because the vertical surface (x_2=0) is rigid along the x_1 direction, it behaves for an incident horizontal component of the displacement field like a fixed end, whereas for an incident vertical component of the displacement field it behaves like a free end. For this reason, the horizontal component of the displacement is reflected at the vertical surface with a reversal of polarity. On the other hand the vertical component of the displacement is reflected without any change of polarity.

At the the corner (the point x_1=0.0km, x_2=0.0km) the waves with only horizontal surface interactions constitute the incident field while the waves interactions with both surfaces constitute the reflected field. Thus, the Green's function components G_{11} and G_{12}, which are the horizontal components of the displacement field, are equal but opposite in sign for the field with horizontal surface interaction only (graph one in Figures 4.7-4.8) and the field with interactions with both surfaces (graph one in Figures 4.11-4.12). The total field for these two components of the Green's function is zero (graph one in Figures 4.15-4.16). The Green's function components G_{21} and G_{22}, which are the vertical components of the displacement field, show the same graphs for both, the field with horizontal surface interactions only (graph one in Figures 4.9-4.10) and the field with interactions with both surfaces (graph one in Figures 4.13-4.14). For this reason the total field for these two components of the Green's function is just double

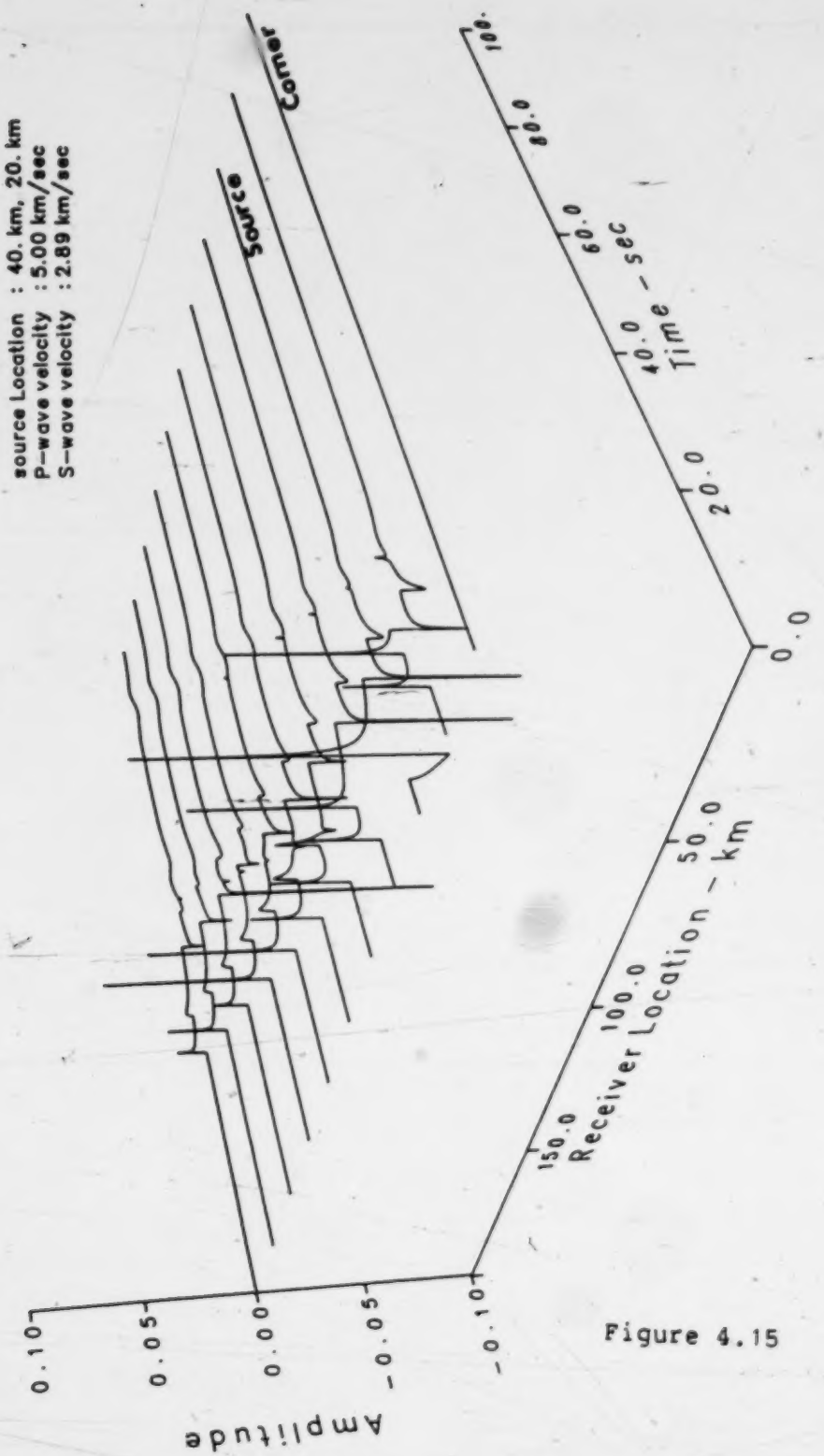
GREEN'S FUNCTION (1,1)
Total Field
source Location : 40. km, 20. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Source
Corner
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Receiver Location - km
0.0
50.0
100.0
150.0
Time - sec
0.0
20.0
40.0
60.0
80.0
100.

Figure 4.15

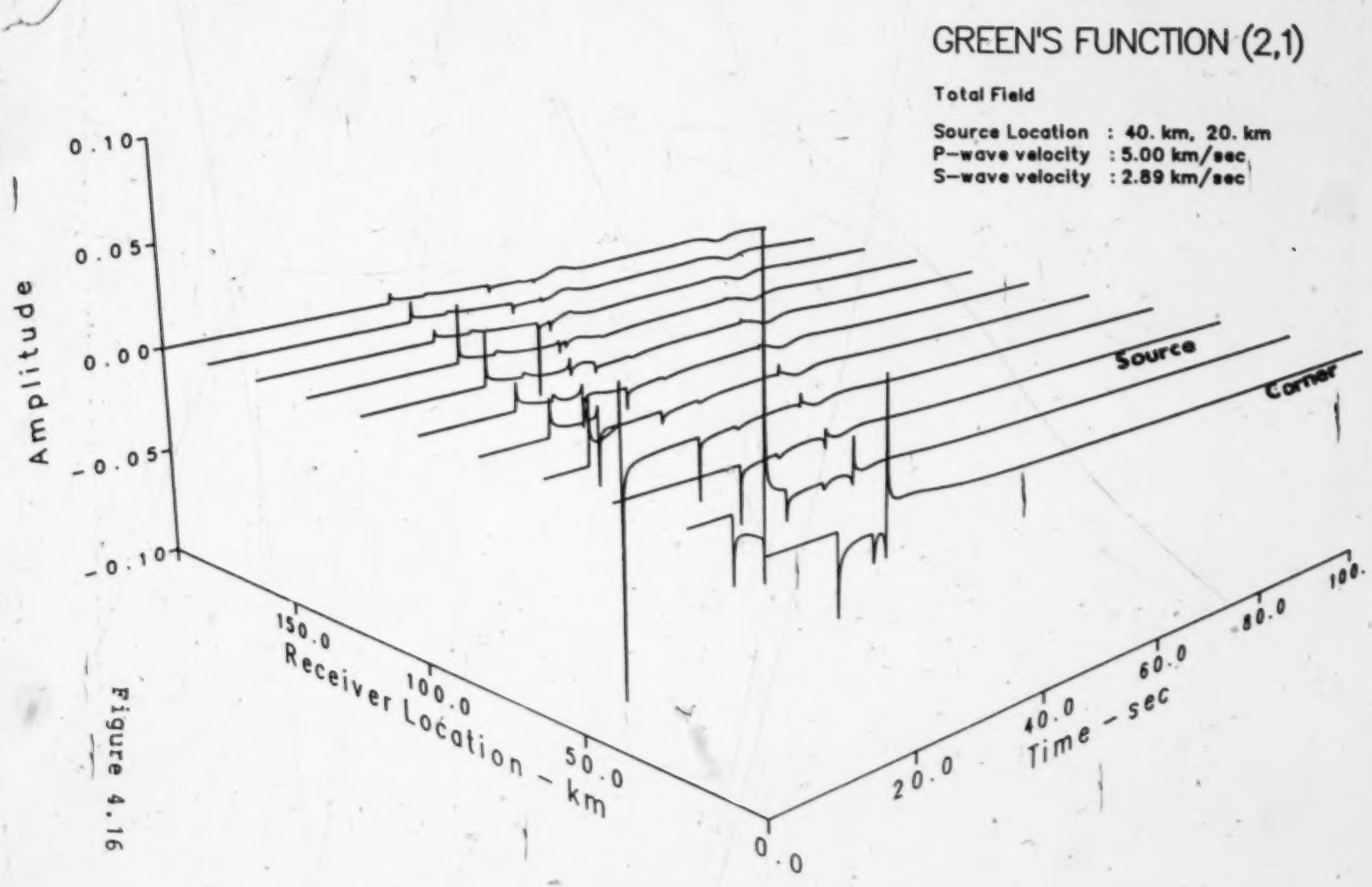
GREEN'S FUNCTION (2,1)
Total Field
Source Location : 40. km, 20. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Source
Corner
150.0
100.0
50.0
0.0
Receiver Location - km
20.0
40.0
60.0
80.0
100.
Time - sec

Figure 4.16

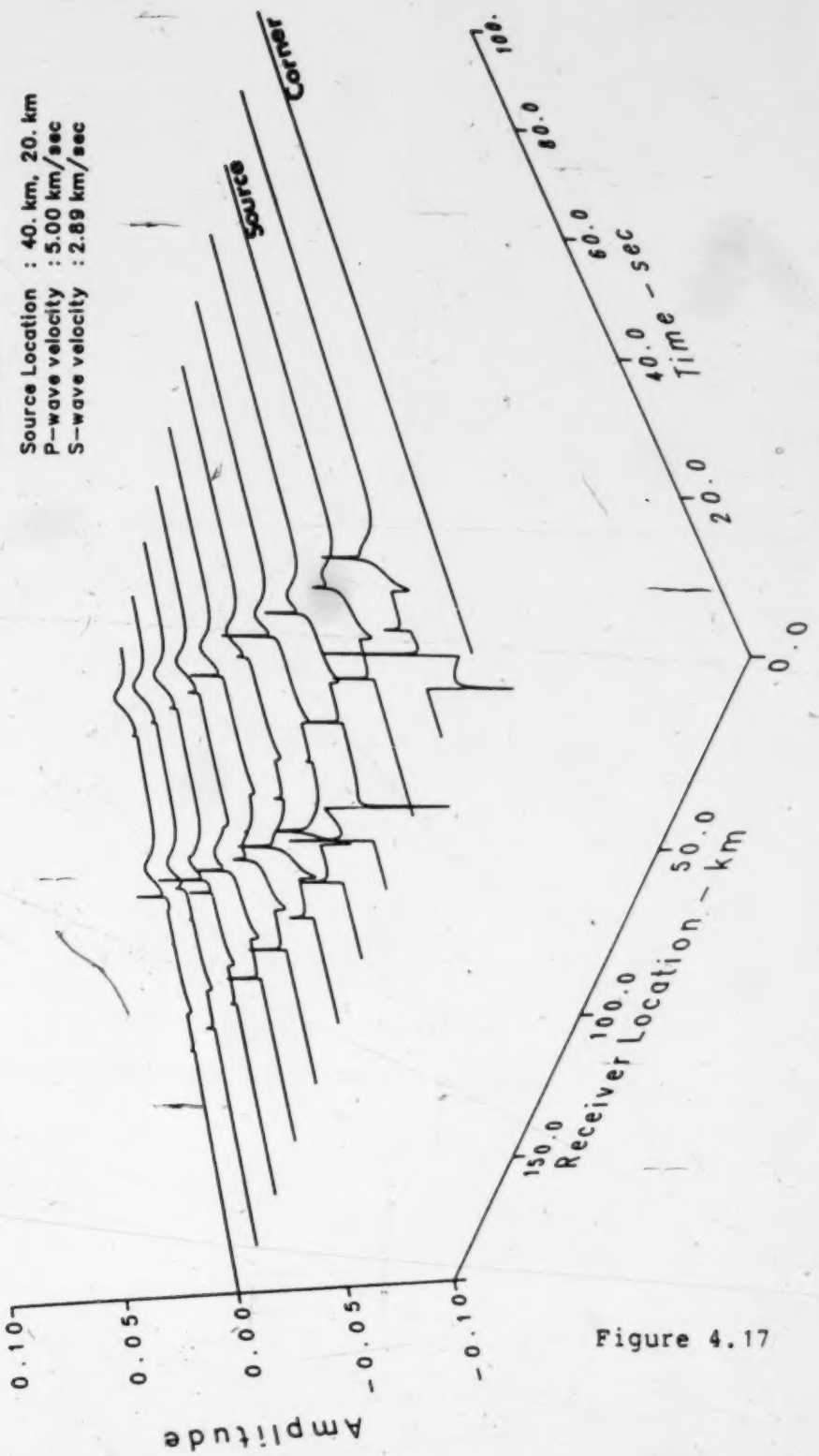
GREEN'S FUNCTION (1,2)
Total Field
Source Location : 40. km, 20. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Source
Corner
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Receiver Location - km
0.0
50.0
100.0
150.0
Time - sec
0.0
20.0
40.0
60.0
80.0
100.

Figure 4.17

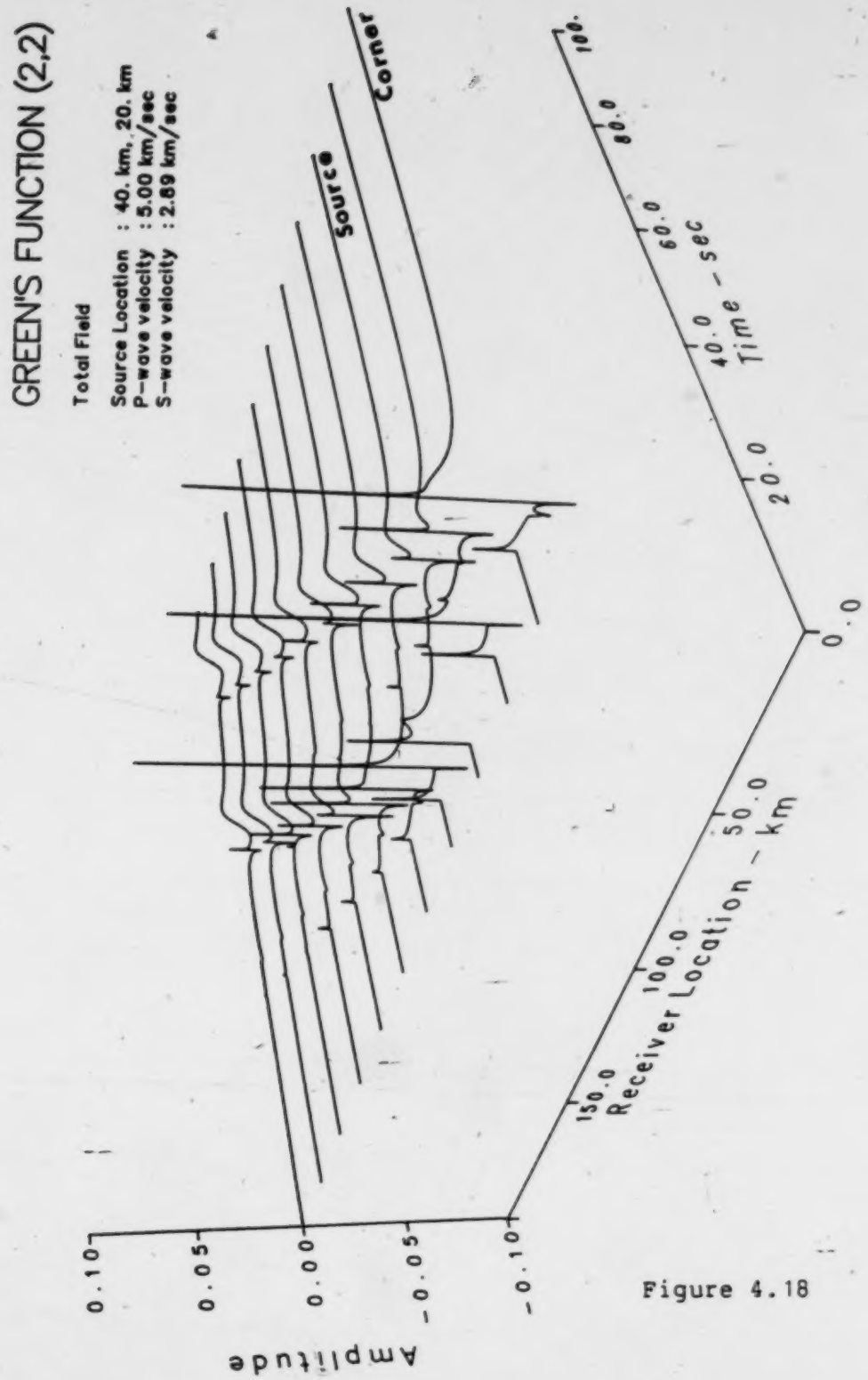
GREEN'S FUNCTION (2,2)
Total Field
Source Location : 40. km, 20. km
P-wave velocity : 5.00 km/sec
S-wave velocity : 2.89 km/sec
Source
Corner
Amplitude
0.10
0.05
0.00
-0.05
-0.10
Receiver Location - km
0.0
50.0
100.0
150.0
Time - sec
0.0
20.0
40.0
60.0
80.0
100.

Figure 4.18

(graph one in Figures 4.17-4.18) that of the individual fields.

For the chosen set of data the critical region is roughly 39km. So, the receiver located at x_1=40km, x_2=0.0km, i.e. directly above the source, as well as the next nearest neighbouring receivers do not show head waves for the field with only horizontal surface interactions. The minimum distance at which the field with interactions with both surfaces occurs is the corner, which is beyond the critical region. Thus, at all receiver locations all four pulses the P , S , head wave and Rayleigh wave are observed for a field interacting with both surfaces.

Directly above the source there is no vertical (horizontal) component of the displacement field due to a force acting in the horizontal (vertical) direction. Indeed G_{21} and G_{12} vanish at a receiver located above the source, for the field with only a horizontal surface interaction. The response of a receiver directly above the source is presented in graph three in Figures (4.7-4.18).

In this computation, a coarse sampling interval was used. Thus the peak amplitudes are somewhat uncertain.

5. Conclusion

We have demonstrated an eigenfunction expansion approach to a quarterspace problem. The results obtained here can be used for interpreting seismic data obtained near a vertical contact of a sedimentary basin with a metamorphic regime. Using this formulation one can estimate the effect of the vertical contact on the observed data. By subtracting such contributions from the observed data one can eliminate the effect of the vertical contact and reduce the problem to a form suitable for standard data processing.

A similar calculation may be carried out in the cylindrical coordinate system to develop the solution for the elastodynamics of a cylindrical hole in a halfspace. This result will be helpful for interpreting borehole seismic data.

Next in the hierarchy of complexity of nonparallel boundary problems, a quarterspace with one boundary rigid and the other stress-free may be studied. Here, mode conversion at both boundaries and the excitation of evanescent waves at only one surface have to be considered. The eigenfunctions developed in this thesis may be used as an "ansatz" for this case. Subsequently one may use these eigenfuntions for developing the eigenfunctions for a quarterspace with both surfaces stress-free.

With the experience gained with such problems the method can be extended in order to obtain a complete set of eigenfunctions for: i) a semi-infinite plate ii) a layered

quarterspace and iii) two quarterspaces in contact. These problems can provide an insight into the solution of more realistic geological problems, such as layered media with normal faults.

One could also apply approximate theories such as asymptotic ray theory or finite difference methods to these problems. But the results obtained by such methods do not necessarily preserve the reciprocity relation. On the other hand, the hermiticity of the wave operator suggests that the symmetry of the Green's function for these problem (i.e. the reciprocity relation) should be preserved by the solution.

In the method outlined in this thesis, the full elastic field is expanded in terms of a complete set of orthonormal functions, so that the reciprocity is ensured for each term of the expansion. Hence, even for an approximate elastic field obtained by keeping only a few terms of the expansion, the reciprocity relation will still be preserved.

On the other hand, in the case of approximate theories such as asymptotic ray theory, the elastic field is obtained by taking only the first few terms of an asymptotic expansion. Here the individual terms of the expansion are not orthonormal, and therefore the reciprocity relation is not preserved. Razavy and Lenoach (1986) have also shown explicitly that the reciprocity property is not necessarily preserved by asymptotic ray theory or by finite difference methods.

Since the reciprocity relation provides a check for the validity of the solution and the accuracy of the synthetic seismograms, the preservation of this property is desirable, especially in the absence of experimental data necessary to verify the theoretical results.

The eigenfunctions developed for nonparallel geometries could have application for the study of eletron mobility in solid surfaces (Ezawa (1971)). In many instance solid surfaces in question are expressible only in terms nonparallel geometries. To study this problem, quantization of elastic waves, in solid which have boundary surfaces, has to be carried out. For this purpose, a complete set of eigenfunctions for the corresponding elastodynamics problem is required. Thus the study of the elastodynamics of nonparallel boundary-value problems is also important for the study of quantum physics of solid surfaces.

References

Alsop, L. E., 1966. Transmission and Reflection of Love waves at a vertical discontinuity, Jour. of Geophys. Res., 71, 3969-3984.

Alsop, L. E., Goodman, A. S. and Gregersen, S., 1974. Reflection and Transmission of inhomogeneous waves with particular application of Rayleigh waves, Bull. Seism. Soc. Am., 64, 1635-1652.

Alterman, Z., S. and Rotenberg, A., 1969. Seismic waves in a quarter plane, 59, 347-368.

Aki, K. and Richards, P. G., 1980. Quantitative Seismology Theory and Methods, Vol. 1, W. H. Freeman and Company, San Fransisco.

Cagniard, L., 1939. Reflexion et réfraction des ondes séismiques progressives, Gauthiér-Villars, Paris (Reflection and Refraction of Progressive Seismic waves, tr. Flinn, E. A. and Dix, C. H., 1962. McGraw Hill, New York.).

Capon, J., 1970. Analysis of Rayleigh waves multipath propagation at LASA, Bull. Seism. Soc. Am., 60, 1701-1731.

Chen, T. C. and Alsop, L. E., 1979. Reflection and transmission of obliquely incident Rayleigh waves at vertical discountinuity between two welded quaterspace, Bull. Seism. Soc. Am., 69, 1409-1423.

de Bremaecker, J. Cl., 1958. Transmission and Reflection of Rayleigh waves at corner, Geophysics, 23, 253-266.

Drake, L. A., 1972A. Love and Rayleigh waves in nonhorizontal layered media, Bull. Seism. Soc. Am., 62, 1241-1250.

Drake, L. A., 1972B. Rayleigh waves at continental boundary by finite element method, Bull. Seism. Soc. Am., 62, 1259-1268.

Ets, E. N. and Yanovskaya, T. B., 1985. Propagation of Surface waves in a half space with vertical, inclined or curved interfaces, Wave Motion, 7, 79-94.

Ezawa, H., 1971. Phonons in a Half Space, Annals of Physics, 67, 428-460.

Forristall, G. Z. and Ingram, J. D., 1971. Elastodynamics of a Wedge, Bull. Seism. Soc. Am., 61, 275-287.

Gregersen, S., 1978. Possible mode conversion between Love and Rayleigh waves at continental margin, Geophys. Jour. Roy. Astr. Soc., 54, 121-127.

Gregersen, S. and Alsop, L. E., 1974. Amplitude of horizontally refracted Love waves, Bull. Seism. Soc. Am., 64, 535-553.

Gregersen, S. and Alsop, L. E., 1976. Mode conversion of Love waves at a continental margin, Bull. Seism. Soc. Am., 66, 1855-1872.

Hong, T. L. and Helmberger, D. V., 1977. Generalized ray theory for dipping layer, Bull. Seism. Soc. Am., 67, 995-1008.

Hudson, J. A., 1963. SH-waves in a wedge-shaped medium, Geophys. Jour. Roy. Astr. Soc., 7, 517-546.

Hudson, J. A. and Knopoff, L., 1964A. Transmission and Reflection of surface waves at a corner 1. Love waves (Theoretical), Jour. of Geophys. Res., 69, 275-280.

Hudson, J. A. and Knopoff, L., 1964B. Transmission and Reflection of surface waves at a corner 2. Rayleigh waves (Theoretical), Jour. of Geophys. Res., 69, 281-289.

Kane, J. and Spence, J., 1963. Rayleigh waves transmission on elastic wedge, Geophysics, 28, 715-723.

Knopoff, L., 1969. Elastic wave propagation in a wedge, In: Wave propagation in solids, ed. Miklowitz, J., Am. Soc. of Mechanical Engineers, 3-43.

Knopoff, L. and Gangi, A. F., 1960. Transmission and Reflection of Rayleigh waves by wedges, Geophysics, 25, 1203-1214.

Lamb, H., 1904. On the propagation of tremors over the surface of an elastic solid, Philos. Trans. Roy. Soc. London, A203, 1-42.

Lapwood, E. R., 1961. The transmission of a Rayleigh pulse round a corner, Geophys. Jour. Roy. Astr. Soc., 4, 174-196.

Lopez Soto, L. A., 1967. Propagation de ondas de Rayleigh en cuñas elastics, Publication #47, Dept. of Geophysics and Geodesy, Univ. of Chile, Santiago.

Lysmer, J. and Drake, L. A., 1971. The propagation of Love waves across nonhorizontally layered structures, Bull. Seism. Soc. Am., 61, 1233-1251.

Mal, A. K. and Knopoff, L., 1966. Transmission of Rayleigh waves at a corner, Bull. Seism. Soc. Am., 56, 455-466.

McGarr, A. and Alsop, L. E., 1967. Transmission and Reflection of Rayleigh waves at vertical boundaries, Jour. of Geophys. Res., 72, 2169-2180.

Miklowicz, J., 1982. Wavefront analysis in the nonseparable elastodynamic quarter-plane problems, Jour. Appl. Mech., 104, 797-815.

Pilant, W. L., Knopoff, L. and Schwab, F. A., 1964. Transmission and Reflection of surface waves at a corner 3. Rayleigh waves (Experimental), Jour. of Geophys. Res., 69, 291-297.

Pao, Y.-H. and Zeigler , F., 1982. Transient SH-waves in a wedge-shaped layer, Geophys. Jour. Roy. Astr. Soc., 71, 57-77.

Razavy, M. and Lenoach, B., 1986. Reciprocity principle and the approximate solution of the wave equation, to appear in December 1986 issue of Bull. Seism. Soc. Am..

Sikharulidze, D. I., 1969. Exprimental study of reflected surface waves during an earthquake, Phys. of the Solid Earth, 5, 274-279.

Singh, S. J., Ben-Menahem, A. and Vered, M., 1968. A unified approach to the representation of seismic sources, Proc. Roy. Soc. Lond., A 331, 525-551.

Sommerfeld, A., 1896. Mathematische theorie der diffraction, Math. Ann., 47, 317-374.

Stokes, G. G., 1849. On the dynamical theory of diffraction, Trans. Cambridge Philo. Soc., 9, 1- 62.

Viswanathan K., Kuo J. T. and Lapwood E. R., 1971. Reflection and transmission of Rayleigh waves in a wedge I, Geophys. Jour. Roy. Astr. Soc., 24, 401-414.

Zeigler , F. and Pao, Y.-H., 1984. Transient elastic waves in a wedge shaped layer, Acta Mechanica, 52, 133-163.

Zeigler , F. and Pao, Y.-H., 1985. Theory of Generalized Rays for SH-waves in dipping layers, Wave Motion, 7, 1-24.

Appendix I

The potentials for the damped modes of type I are

$$\begin{pmatrix} \phi^{(3)} \\ \psi^{(3)} \end{pmatrix} = a_s \begin{pmatrix} \mathrm{Cos}kx_1 \; e^{-\Gamma x_2} \\ \mathrm{Sin}kx_1 \left\{ \frac{2\,k\,\Gamma}{(\beta^2 - k^2)} \mathrm{Cos}\beta x_2 - \frac{(\beta^2 - k^2)}{2\,k\,\beta} \mathrm{Sin}\beta x_2 \right\} \end{pmatrix}$$

$$\Theta(k-K_p)\;\Theta(K_s-k)\;. \qquad (I.1)$$

Now we have to fix $a_s(k,\omega)$ using the identity

$$\begin{aligned} \langle \vec{u}^{(3)}, \vec{u}^{(3)'} \rangle &= \langle \nabla\phi^{(3)}, \nabla\phi^{(3)'} \rangle + \langle \nabla\phi^{(3)}, \nabla\times\vec{1}_3\psi^{(3)'} \rangle \\ &+ \langle \nabla\times\vec{1}_3\psi^{(3)}, \nabla\phi^{(3)'} \rangle \\ &+ \langle \nabla\times\vec{1}_3\psi^{(3)}, \nabla\times\vec{1}_3\psi^{(3)'} \rangle \\ &= \delta(k-k')\;\delta(\omega-\omega')\;. \qquad (I.2) \end{aligned}$$

Here the prime indicates that k' and ω' instead of k and ω in the arguments.

We find first

$$\langle \nabla\phi^{(3)}, \nabla\phi^{(3)'} \rangle = \int_0^\infty dx_1 \int_0^\infty dx_2, \nabla\phi^{(3)*} \cdot \nabla\phi^{(3)'}$$

$$= \int_0^{\infty} dx_1 \int_0^{\infty} dx_2 \; a_s^* a_s' \{kk' \; \mathrm{Sin}kx_1 \; \mathrm{Sin}k'x_1$$

$$+ \Gamma\Gamma' \; \mathrm{Cos}kx_1 \; \mathrm{Cos}k'x_1\} \; e^{-(\Gamma+\Gamma')x_2}$$

$$= \frac{\pi}{2} \delta(k-k') \; a_s^* a_s' \frac{k^2 + \Gamma\Gamma'}{\beta^2 + \Gamma'} \; . \qquad (I.3)$$

Similarly we have,

$$\langle \nabla\phi^{(3)}, \nabla\times\vec{1}_3\psi^{(3)'} \rangle = \frac{\pi}{2} \delta(k-k') \; a_s^* a_s' \frac{2 \; k^2 \; \Gamma'}{(\beta'^2 - k^2)} \; , \qquad (I.4)$$

$$\langle \nabla\times\vec{1}_3\psi^{(3)}, \nabla\phi^{(3)'} \rangle = \frac{\pi}{2} \delta(k-k') \; a_s^* a_s' \frac{2 \; k^2 \; \Gamma}{(\beta^2 - k^2)} \; , \qquad (I.5)$$

and

$$\langle \nabla\times\vec{1}_3\psi^{(3)}, \nabla\times\vec{1}_3\psi^{(3)'} \rangle = \frac{\pi^2}{4} \delta(k-k') \; \delta(\beta-\beta') \; a_s^* a_s' \; (\beta^2 + k^2)$$

$$\left[\frac{4 \; k^2 \; \Gamma^2}{(\beta^2 - k^2)^2} + \frac{(\beta^2 - k^2)^2}{4 \; k^2 \; \beta^2} \right]$$

$$+ \frac{\pi}{2} \delta(k-k') \; a_s^* a_s' \frac{1}{(\beta^2 - \beta'^2)^2}$$

$$\left[\frac{\Gamma}{(\beta^2 - k^2)} (\beta'^2 - k^2)(\beta^2 + k^2) \right.$$

$$\left. - \frac{\Gamma'}{(\beta'^2 - k^2)^2} (\beta^2 - k^2)(\beta'^2 + k^2) \right] . \qquad (I.6)$$

Collecting all terms in equations (I.3)-(I.6) do not contain that delta functions, we have

$$\frac{(k^2+\Gamma\Gamma')}{(\Gamma+\Gamma')}+\frac{\Gamma}{(\beta^2-k^2)}\left[2k^2+\frac{(\beta'^2-k^2)(\beta^2+k^2)}{(\beta^2-\beta'^2)^2}\right]$$

$$+\frac{\Gamma'}{(\beta'^2-k^2)^2}\left[2k^2-\frac{(\beta^2-k^2)(\beta'^2+k^2)}{(\beta^2-\beta'^2)}\right].$$

After some algebraic rearrangements this expression simplifies to

$$\frac{(\Gamma^2-k^2)(\beta'^2+k^2)-(\Gamma'^2-k^2)(\beta^2+k^2)}{(\Gamma+\Gamma')(\beta^2-\beta'^2)}.$$

On further simplification it becomes:

$$\frac{-K_p^2K_s'^2+K_p'^2K_s^2}{(\Gamma+\Gamma')(\beta^2-\beta'^2)}$$

$$=\frac{\frac{-\omega\omega'}{C_pC_s}+\frac{\omega'\omega}{C_pC_s}}{(\Gamma+\Gamma')(\beta^2-\beta'^2)}$$

$$=0.$$

Therefore, adding all four terms in equation (I.2), the contributions of the damped parts drop out and we have

$$\langle\vec{u}^{(3)},\vec{u}^{(3)'}\rangle=\frac{\pi^2}{4}\,\delta(k-k')\,\delta(\beta-\beta')\,a_s^*a_s'\times$$

$$\omega\,\frac{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}{4k^2\beta(\beta^2-k^2)^2}. \qquad (I.7)$$

Here the identity $\delta(\beta-\beta')=\delta(\omega-\omega')\,\frac{\omega\beta}{(\beta^2+k^2)}$ was used.

Thus, equation (I.7) yields the normalization constant

$$a_s(k,\omega) = \frac{2}{\pi} \frac{2k(\beta^2 - k^2)\sqrt{\beta/\omega}}{\sqrt{(\beta^2 - k^2)^4 + 16k^4\Gamma^2\beta^2}} . \quad (I.8)$$

Appendix II

Collecting the terms, in equation (3.23) involving $\beta(x_2+x_2')$ and its analytic continuation we have:

$$\frac{2}{\pi^2}\int_0^\infty dk\ \mathrm{Sin}kx_1\,\mathrm{Sin}kx_1'$$

$$\left[\int_{kC_p}^{\infty}\frac{d\omega}{\omega}\,\beta\,\frac{(\beta^2-k^2)^2-4k^2\alpha\beta}{(\beta^2-k^2)^2+4k^2\alpha\beta}\,\mathrm{Cos}\beta(x_2+x_2')\right.$$

$$+\int_{kC_s}^{kC_p}\frac{d\omega}{\omega}\,\beta\,\frac{(\beta^2-k^2)^4-16k^4\Gamma^2\beta^2}{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}\,\mathrm{Cos}\beta(x_2+x_2')$$

$$\left.+\int_{kC_s}^{kC_p}\frac{d\omega}{\omega}\,\beta\,\frac{8k^2\Gamma(\beta^2-k^2)^2\beta}{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}\,\mathrm{Sin}\beta(x_2+x_2')\right]$$

$$+\frac{1}{\pi}\int_0^\infty dK_r\ \mathrm{Sin}K_rx_1\,\mathrm{Sin}K_rx_1'\,\frac{(\Delta_r^2+K_r^2)^2}{R(K_r)}\,\Delta_r\,\frac{4K_r^2\Gamma_r^2\Delta_r^2}{(\Delta_r^2+K_r^2)^2}\cdot$$

$$e^{-\Delta_r(x_2+x_2')}\ . \qquad (II.1)$$

This expression can be rewritten as follows:

$$\frac{1}{\pi^2}\int_0^\infty dk\ \mathrm{Sin}kx_1\,\mathrm{Sin}kx_1'\left[\int_{kC_p}^{\infty}\frac{d\omega}{\omega}\,\beta\,\frac{(\beta^2-k^2)^2-4k^2\alpha\beta}{(\beta^2-k^2)^2+4k^2\alpha\beta}\right.$$

$$\left.\left[e^{i\beta(x_2+x_2')}+e^{-i\beta(x_2+x_2')}\right]\right.$$

$$+ \int_{kC_s}^{kC_p} \frac{d\omega}{\omega}\,\beta \left[\frac{(\beta^2-k^2)^2-4k^2i\Gamma\beta}{(\beta^2-k^2)^2+4k^2i\Gamma\beta}\, e^{i\beta(x_2+x_2')} + \frac{(\beta^2-k^2)^2+4k^2i\Gamma\beta}{(\beta^2-k^2)^2-4k^2i\Gamma\beta}\, e^{-i\beta(x_2+x_2')} \right]$$

$$+ \frac{1}{\pi}\int_0^\infty dK_r\, \mathrm{Sin}K_r x_1\, \mathrm{Sin}K_r x_1' \frac{(\Delta_r^2+K_r^2)^2}{R(K_r)}\,\Delta_r \frac{4K_r^3\Gamma_r^3\Delta_r^2}{(\Delta_r^2+K_r^2)^2}$$

$$e^{-\Delta_r(x_2+x_2')} . \qquad \text{(II.2)}$$

As in section 3.3 (see equation (3.27)), we extend ω to the complex ξ plane and combine the first four terms of equation (II.2) which read

$$\frac{1}{2\pi^2}\int_0^\infty dk\, \mathrm{Sin}kx_1\, \mathrm{Sin}kx_1' \int_\Lambda \frac{d\xi}{\xi}\,\eta\, \frac{(\eta^2-k^2)^2-4k^2\zeta\eta}{(\eta^2-k^2)^2+4k^2\zeta\eta}$$

$$e^{i\eta(x_2+x_2')} . \qquad \text{(II.3)}$$

We evaluate the internal integral as before, using the residue theorem. We get:

$$2\pi i\,\frac{1}{2\pi^2}\left[-\frac{-i8K_r^3\Gamma_r\Delta_r^2}{4\,R(K_r)}\,\Gamma_r\Delta_r \right] 2\, e^{-\Delta_r(x_2+x_2')} . \qquad \text{(II.4)}$$

Using the Rayleigh wave secular equation (3.18) this can be rewritten as follows:

$$-\frac{1}{\pi}\left[\frac{(\Delta_r^2+K_r^2)^2}{R(K_r)}\,\Delta_r\right]\frac{2K_r^3\Gamma_r\Delta_r}{(\Delta_r^2+K_r^2)^2}\, e^{-\Delta_r(x_2+x_2')} \qquad \text{(II.5)}$$

which is exactly the negative of the remaining Rayleigh term

in equation (II.2).

Similarly collecting the terms involving $(\alpha x_2+\beta x_2')$ and their analytic continuations from equation (3.23) we have:

$$\frac{2}{\pi^2}\int_0^\infty dk\ \mathrm{Sin}kx_1\ \mathrm{Sin}kx_1'$$

$$\left[\int_{kC_p}^{\infty}\frac{d\omega}{\omega}\,\beta\,\frac{4k^2(\beta^2-k^2)}{(\beta^2-k^2)^2+4k^2\alpha\beta}\,\mathrm{Cos}(\alpha x_2+\beta x_2')\right.$$

$$+\int_{kC_s}^{kC_p}\frac{d\omega}{\omega}\,\beta\left[\frac{16k^4\Gamma\beta(\beta^2-k^2)}{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}\,e^{-\Gamma x_2}\,\mathrm{Sin}\beta x_2'\right.$$

$$\left.\left.+\frac{4k^2(\beta^2-k^2)^3}{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}\,e^{-\Gamma x_2}\,\mathrm{Cos}\beta x_2'\right]\right]$$

$$-\frac{1}{\pi}\int_0^\infty dK_r\ \mathrm{Sin}K_r x_1\ \mathrm{Sin}K_r x_1'\,\frac{(\Delta_r^2+K_r^2)^2}{R(K_r)}\,\Delta_r$$

$$\frac{2K_r^2\Gamma_r\Delta_r}{(\Delta_r^2+K_r^2)}\,e^{-(\Delta_r x_2+\Delta_r x_2')}\ . \qquad \text{(II.6)}$$

Rewriting this expression we have:

$$\frac{1}{\pi^2}\int_0^\infty dk\ \mathrm{Sin}kx_1\ \mathrm{Sin}kx_1'\left[\int_{kC_p}^{\infty}\frac{d\omega}{\omega}\,\beta\,\frac{4k^2(\beta^2-k^2)}{(\beta^2-k^2)^2+4k^2\alpha\beta}\right.$$

$$\left[e^{i(\alpha x_2+\beta x_2')}+e^{-i(\alpha x_2+\beta x_2')}\right]$$

$$+\int_{kC_s}^{kC_p}\frac{d\omega}{\omega}\,\beta\left[\frac{4k^2(\beta^2-k^2)}{(\beta^2-k^2)^2+4k^2i\Gamma\beta}\right.$$

$$e^{-(\Gamma x_2+i\beta x_2')} + \frac{4k^2(\beta^2-k^2)}{(\beta^2-k^2)^2-4k^2i\Gamma\beta} e^{-(\Gamma x_2-i\beta x_2')}\Bigg]\Bigg]$$

$$-\frac{1}{\pi}\int_0^\infty dK_r \, \mathrm{Sin}K_r x_1 \, \mathrm{Sin}K_r x_1' \frac{(\Delta_r^2+K_r^2)^2}{R(K_r)} \Delta_r$$

$$\frac{2K_r^2\Gamma_r\Delta_r}{(\Delta_r^2+K_r^2)} e^{-(\Delta_r x_2+\Delta_r x_2')} . \qquad \text{(II.7)}$$

As before, we consider the complex ξ plane and combine the first four terms of equation (II.7) to obtain:

$$\frac{1}{2\pi^2}\int_0^\infty dk \, \mathrm{Sin}kx_1 \, \mathrm{Sin}kx_1' \int_\Lambda \frac{d\xi}{\xi} \eta \frac{4k^2(\eta^2-k^2)}{(\eta^2-k^2)^2+4k^2\zeta\eta}$$

$$e^{i(\zeta x_2+\eta x_2')} . \qquad \text{(II.8)}$$

Using the residue theorem we find this integral to be equal to

$$2\pi i \frac{1}{2\pi^2}\left[\frac{-i4K_r^2(\Delta_r^2-k^2)\Gamma_r\Delta_r^2}{4R(K_r)} \Gamma_r\Delta_r\right] 2 e^{-(\Gamma_r x_2+\Delta_r x_2')} . \qquad \text{(II.9)}$$

Using the Rayleigh wave secular equation (3.18) this can now be further rewritten as follows:

$$\frac{1}{\pi}\left[\frac{(\Delta_r^2+K_r^2)^2}{R(K_r)} \Delta_r\right]\frac{2K_r^2\Gamma_r\Delta_r}{(\Delta_r^2+K_r^2)} e^{-\Delta_r(x_2+x_2')} \qquad \text{(II.10)}$$

This expression, cancels the Rayleigh term part of equation

(II.7)

By interchanging x_2 and x_2' in the above expressions related to $(\alpha x_2+\beta x_2')$, we obtain the expressions for the $(\alpha x_2'+\beta x_2)$ terms.

Appendix III

For the sake of completeness, expressions for the off-diagonal components of the Green's function are presented here.

The component G_{12} may be expressed as:

$$G_{12}(\vec{x},\vec{x}',\tau) = \int_{-\infty}^{+\infty} dp\, e^{-ip\tau} \sum_{n} \int dk\, d\omega \; \frac{u_1^{(n)}(\vec{x},k,\omega)\, u_2^{(n)}(\vec{x}',k,\omega)}{(\omega^2 - p^2)} \,. \qquad (III.1)$$

Proceeding as in section 4.3 this expression is written explicitly, using the eigenfunctions defined in chapter 3, as:

$$G_{12}(\vec{x},\vec{x}',\tau) = \frac{2}{\pi^2} \int_{-\infty}^{+\infty} e^{-ip\tau} \left[\int_0^{\infty} dk\, \mathrm{Sin}kx_1\, \mathrm{Cos}kx_1' \right.$$

$$\left. - \int_{kC_p}^{\infty} \frac{d\omega}{\omega[\omega^2 - (p+i\epsilon)^2]}\, k\, \mathrm{Sin}\alpha(x_2 - x_2') \right]$$

$$+ \int_{kC_s}^{\infty} \frac{d\omega}{\omega[\omega^2 - (p+i\epsilon)^2]}\, k\, \mathrm{Sin}\beta(x_2 - x_2')$$

$$- \int_{kC_p}^{\infty} \frac{d\omega}{\omega[\omega^2 - (p+i\epsilon)^2]}\, k\, \frac{(\beta^2 - k^2)^2 - 4k^2\alpha\beta}{(\beta^2 - k^2)^2 + 4k^2\alpha\beta}\, \mathrm{Sin}\alpha(x_2 + x_2')$$

$$+\int_{kC_s}^{kC_p}\frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\,k\,\frac{8k^2\Gamma(\beta^2-k^2)^2\beta}{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}\,e^{-\Gamma(x_2+x_2')}$$

$$-\int_{kC_p}^{\infty}\frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\,k\,\frac{(\beta^2-k^2)^2-4k^2\alpha\beta}{(\beta^2-k^2)^2+4k^2\alpha\beta}\,\mathrm{Sin}\beta(x_2+x_2')$$

$$-\int_{kC_s}^{kC_p}\frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\,k\,\frac{(\beta^2-k^2)^4-16k^4\Gamma^2\beta^2}{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}\,\mathrm{Sin}\beta(x_2+x_2')$$

$$-\int_{kC_p}^{\infty}\frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\,k\,\frac{4k^2(\beta^2-k^2)}{(\beta^2-k^2)^2+4k^2\alpha\beta}\,\mathrm{Sin}(\alpha x_2+\beta x_2')$$

$$+\int_{kC_p}^{\infty}\frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\,k\,\frac{4\alpha(\beta^2-k^2)\beta}{(\beta^2-k^2)^2+4k^2\alpha\beta}\,\mathrm{Sin}(\beta x_2+\alpha x_2')$$

$$+\int_{kC_s}^{kC_p}\frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\,k\,\frac{1}{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}$$

$$\Big[16k^4\Gamma(\beta^2-k^2)\beta\,e^{-\Gamma x_2}\mathrm{Cos}\beta x_2'-4k^3\Gamma(\beta^2-k^2)^3\,e^{-\Gamma x_2}\mathrm{Sin}\beta x_2'\Big]$$

$$-\int_{kC_s}^{kC_p}\frac{d\omega}{\omega\{\omega^2-(p+i\epsilon)^2\}}\,k\,\frac{1}{(\beta^2-k^2)^4+16k^4\Gamma^2\beta^2}$$

$$\Big[4k^3(\beta^2-k^2)^4\beta\,\mathrm{Cos}\beta x_2 e^{-\Gamma x_2'}+16k^3\Gamma^2(\beta^2-k^2)\beta^2\,\mathrm{Sin}\beta x_2 e^{-\Gamma x_2'}\Big]\Big\}$$

$$+\frac{\pi}{2}\int_0^{\infty}\frac{dK_r}{\{K_r^2C_r^2-(p+i\epsilon)^2\}}\,\mathrm{Sin}K_r x_1\,\mathrm{Cos}K_r x_1'\,\frac{(\Delta_r^2+K_r^2)}{R(K_r)}\,\Delta_r$$

$$\left\{K_r^2\Gamma_r^2\,e^{-\Gamma_r(x_2+x_2')}-\frac{4K_r^2\Gamma_r\Delta_r^2}{(\Delta_r^2+K_r^2)}\,e^{-\Delta_r(x_2+x_2')}\right.$$

$$\left.-\frac{2K_r^3\Delta_r}{(\Delta_r^2+K_r^2)}\,e^{-(\Gamma_r x_2+\Delta_r x_2')}-\frac{2K\Gamma_r^2\Delta_r}{(\Delta_r^2+K_r^2)^2}e^{-(\Delta_r x_2+\Gamma_r x_2')}\right\}\Bigg]. \qquad \text{(III.2)}$$

From equation (4.14) we have the symmetry relation

$$G_{21}(\vec{x},\vec{x}',\tau)=G_{12}(\vec{x}',\vec{x},\tau)\ . \qquad \text{(III.3)}$$

Thus by interchanging $\vec{x}$ and $\vec{x}'$ in equation (III.2) we have the expression for the G_{21} component of the Green's function.